When Combinatorics and Flow Networks Intersect

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G'day!

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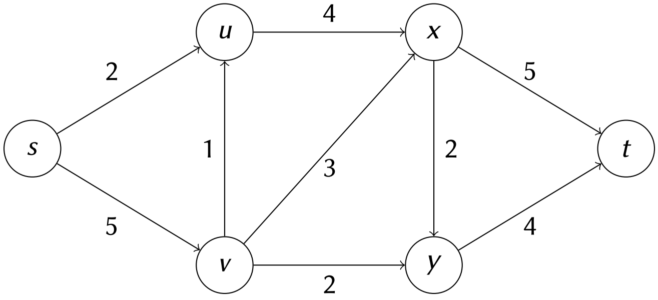
- I like:
  - Ruining my sleep schedule from time to time.
  - Teaching and learning about new things.
- Nom nom.

Introduction to Maximum Flow

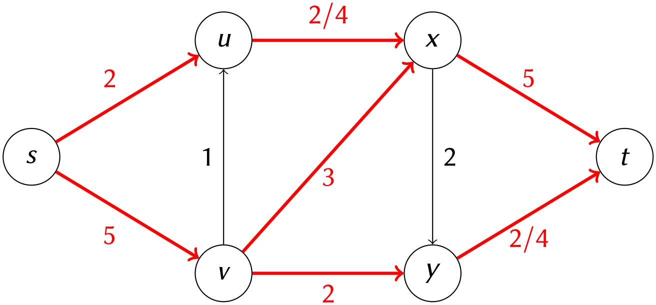
Introduction to Maximum Flow

A **flow network** is a directed and weighted graph $G = (V, E)$, where each edge $(u, v) \in E$ has a weight $w_{u,v}$. This is called the *capacity*.



The Maximum Flow Problem

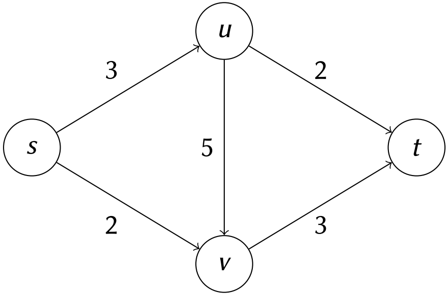
- Given a flow network, how much flow can we send from s to t assuming we have an infinite supply in s?



Maximum flow: 7.

Ford-Fulkerson

- Try as many paths as possible!
- Find $s - t$ paths and send flow down the path.
- When updating flows and capacities, send flow back an edge.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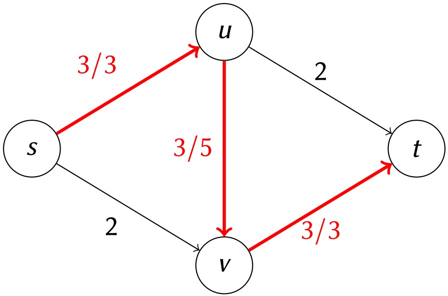
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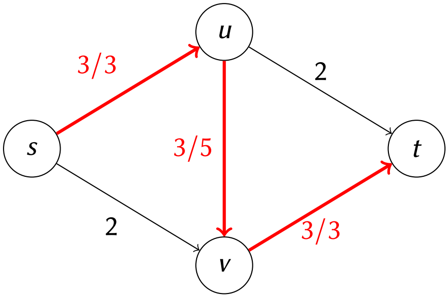
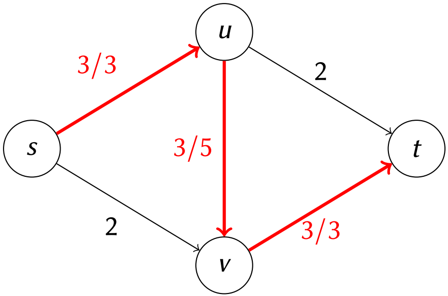
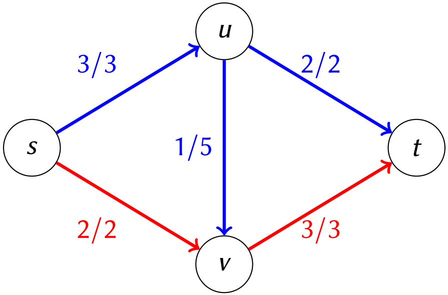


Flow: 3.



Flow: 3. Hmmm... can we do better?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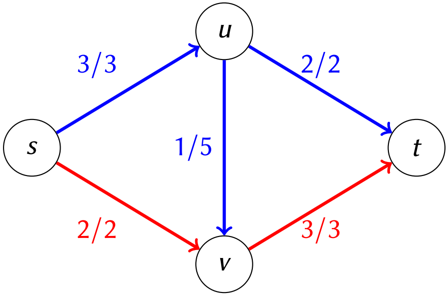
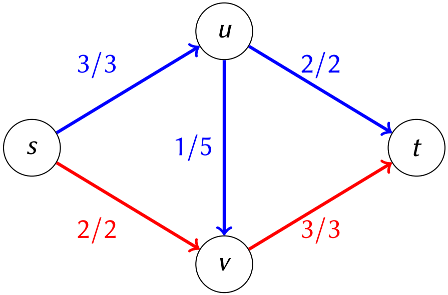


Flow: 5.



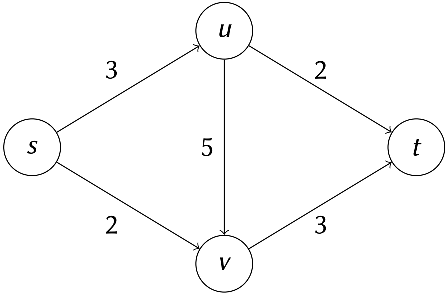
Flow: 5. So... what went wrong?

Ford-Fulkerson

- We need a way to “undo” flow.

Ford-Fulkerson

- We need a way to “undo” flow.
  - We can denote the amount of flow we can send back with an arrow in the *reverse* direction.
  - Keep finding $s - t$ paths this way until no more paths are available.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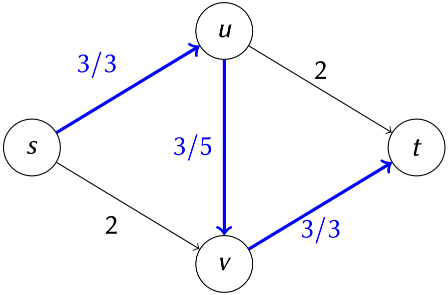
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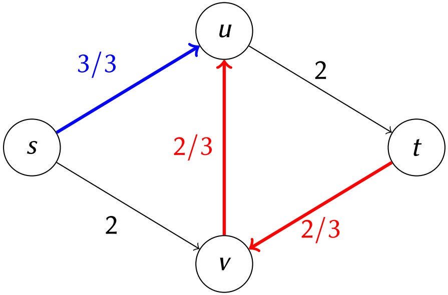
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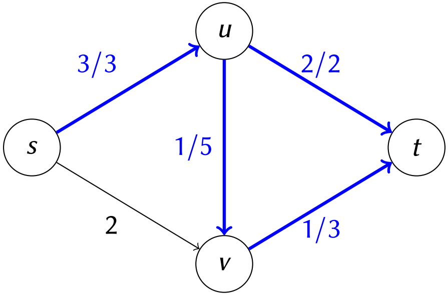
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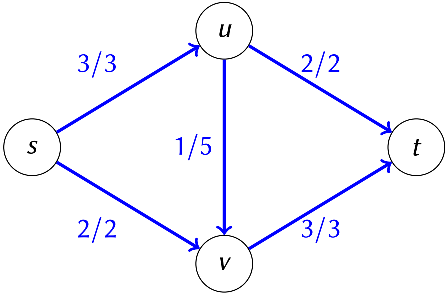
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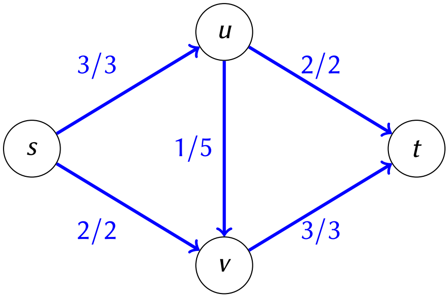

Terminate with maximum flow of 5.

Ford-Fulkerson

- Note that there are finite many paths from s to t; therefore, the algorithm must terminate.
- Every time we "reuse" an edge, we send flow back to try for a better $s - t$ path.
- The final output of the Ford-Fulkerson algorithm is a set of "saturated" edges which correspond to the edges that are used in the maximum flow of the flow network.
- **Running time**: $O(|E| \cdot |f|)$, where $|f|$ is the flow of the graph.

Other algorithms

Other algorithms exist that solve the Maximum Flow problem with various running times.

- Edmonds-Karp – special modification of Ford-Fulkerson: $O(|E| \cdot \min\{|V| \cdot |E|, |f|\})$.
- Dinic's algorithm – $O(|V|^2 \cdot |E|)$.
- Preflow push algorithm – $O(|V|^2 \cdot |E|)$.

Maximum Flow-Minimum Cut

Cuts in a Flow Network

A *cut* in a flow network is a partition of vertices into two sets S and T such that:

- $S \cup T = V$.
- $S \cap T = \emptyset$.
- $s \in S, t \in T$.

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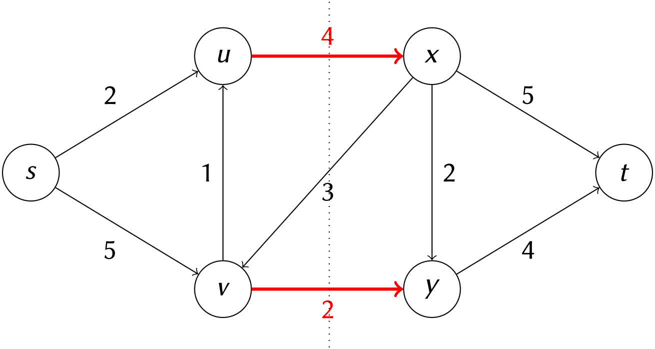
Capacity of a cut

The *capacity of a cut* is the sum of the capacity of the edges that "pass" through the cut in the forward direction (i.e. a directed edge from $u \in S$ *to* $v \in T$).



Capacity of cut: 6.

Maximum Flow-Minimum Cut Theorem

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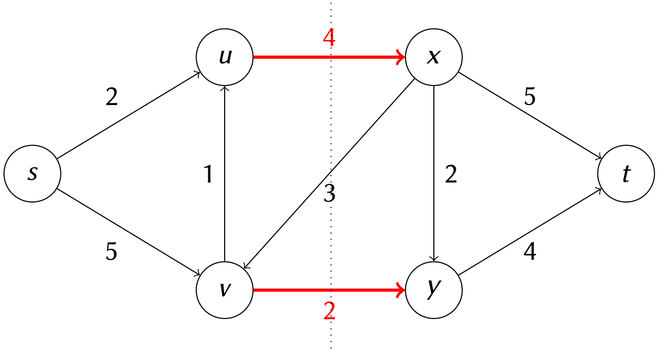
The maximum flow of a flow network corresponds to the minimum capacity cut of the flow network.

Maximum Flow-Minimum Cut Theorem



- All $s - t$ paths must pass through the red edges.
  - Minimum cut – limits the amount of flow that can be sent to these edges.
  - Maximum flow – must send flow along the edges along the minimum cut.

The Combinatorial Results

General structure of the theorems

Given a structure, the maximum of A corresponds to the minimum of B.

Given a flow network F, the maximum flow of F corresponds to the minimum cut of F.

It turns out there are many other theorems that have this same shape!

Hall's Marriage Theorem

Let $\mathcal{F}$ be a family (or *collection*) of sets and let X be the union of elements in all sets of $\mathcal{F}$.

Transversal of a set

We say that a subset $S \subseteq X$ is a *transversal* for $\mathcal{F}$ if S is comprised of one element from each set in $\mathcal{F}$.

In other words, for each set F in $\mathcal{F}$, pick one element from F to represent the set.

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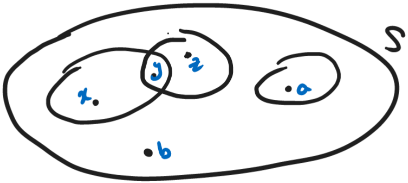

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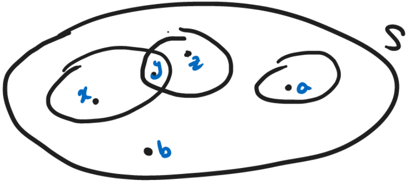

Hmm... assigning an element directly from S might not give us the right assignment because we could accidentally choose an element that doesn't appear in any set in $\mathcal{G}$. Oops...
Let's fix this!

Hall's Marriage Theorem

Let's try again!

When does a transversal exist? Let's consider a subcollection $\mathcal{G}$ of sets in $\mathcal{F}$. We denote Y to be the set of elements that belong to at least one set in $\mathcal{G}$.

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- Assign an element from Y to represent a set in $\mathcal{G}$.

We now have limited our choice of elements to all elements that belong in some set in $\mathcal{G}$. However, what if we don't have enough elements?

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Our theorem!

Hall's Marriage Theorem

Let $\mathcal{F}$ be a family (collection) of finite sets. Then $\mathcal{F}$ has a transversal if and only if, for every subcollection $\mathcal{G} \subseteq \mathcal{F}$,

$$|\mathcal{G}| \leq \left| \bigcup_{S \in \mathcal{G}} S \right|.$$

Reformulating Hall's Marriage Theorem

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    - Therefore, an edge represents the possibility of a married couple.
    - For *any* collection of women, we need to have enough men to match to each woman.
- This forms a bipartite graph, where one partition of vertices represents possible women and the other partition of vertices represents possible men. Every woman can be matched with a man if $|W| \leq |N(W)|$, where W is a set of women and $N(W)$ represents the men that is connected to at least one woman in W.

$$
\begin{aligned}
\mathcal{F} &= \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},\\
\mathcal{A}_1 &= \{a, b, c\},\\
\mathcal{A}_2 &= \{a\},\\
\mathcal{A}_3 &= \{c, d\},\\
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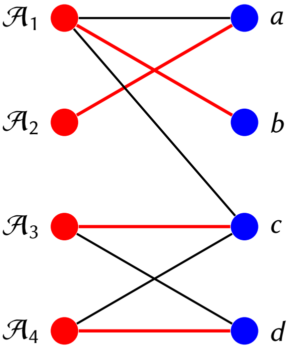


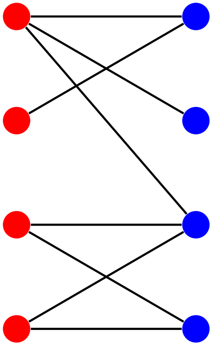
Graph-theoretic formulation of Hall's Marriage Theorem

Hall's Marriage Theorem

More formally, let $G = (V, E)$ be a bipartite graph with partition V_1 and V_2 such that $V_1 \cup V_2 = V$. Also, suppose that $|V_1| = |V_2|$. Then G has a *perfect* matching if and only if, for every $S \subseteq V_1$,

$$|S| \leq |N(S)|.$$

Maximum Flow-Minimum Cut $\implies$ HMT



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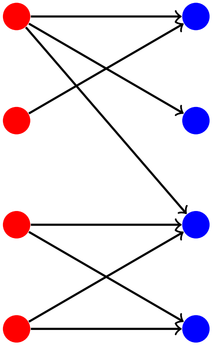

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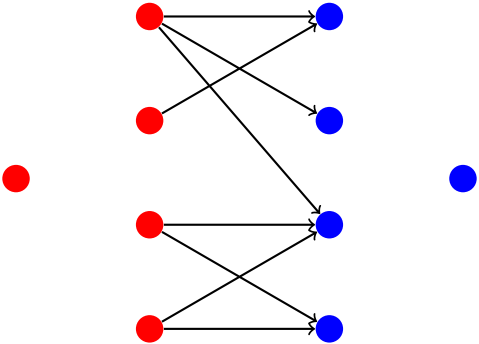

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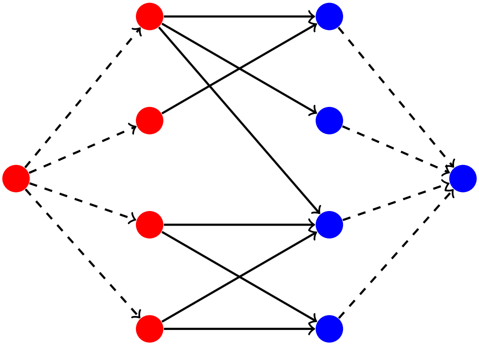

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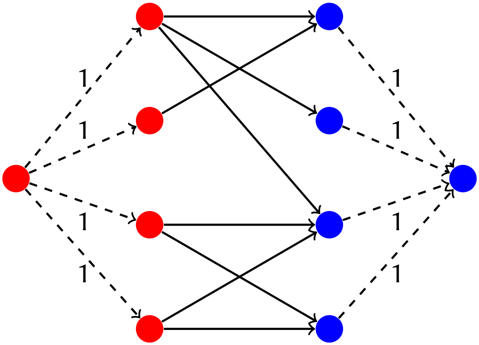

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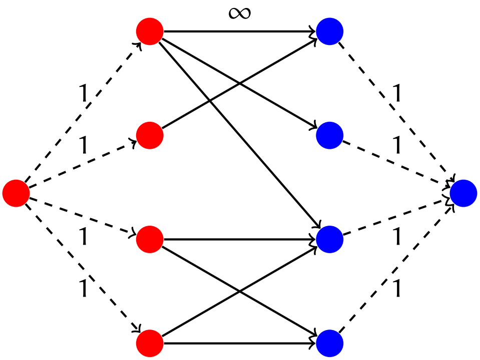

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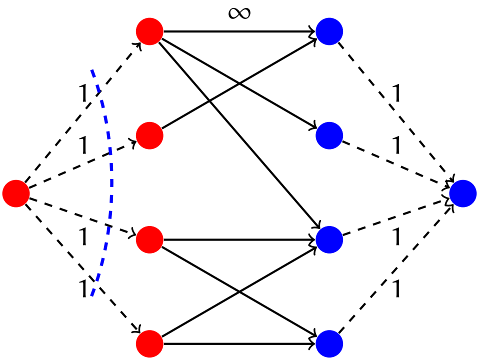

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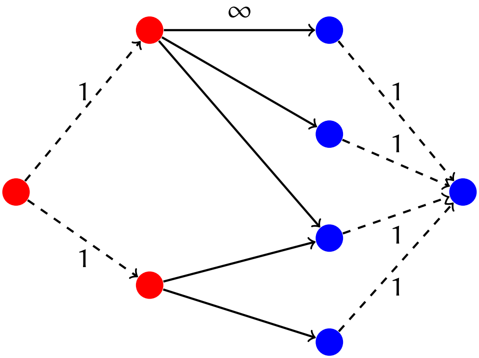

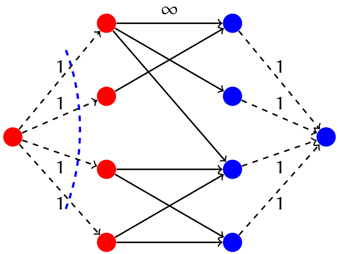

An example of a bipartite graph that satisfies Hall's condition and an example of a bipartite graph that does not satisfy Hall's condition.

- Taking the last two vertices in the red vertex set does not satisfy Hall's condition. Note that the maximum flow of the second flow network is 3.

Dilworth's Theorem

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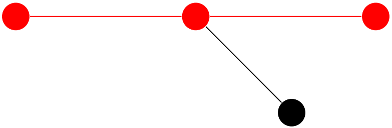
Chains of P

Let P be a finite partially ordered set. A chain is a subset $C \subseteq P$ such that, for any two elements $x, y \in C$, either $R(x, y)$ or $R(y, x)$. We say that x and y are *comparable*.



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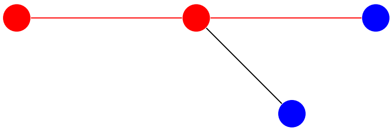

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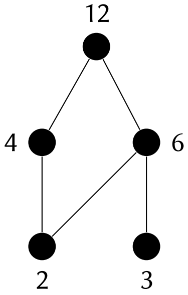

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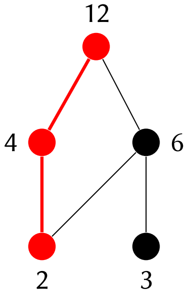

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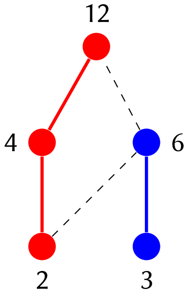

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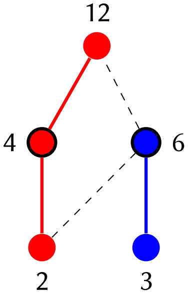

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It turns out that the largest sized antichain corresponds to the smallest number of chains required to cover P! This is our theorem that we want to explore.

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Dilworth's Theorem

Let P be a finite partially ordered set and suppose that C is the smallest collection of disjoint chains that partition P. Let $\mathcal{A}$ be a largest antichain of P. Then $|\mathcal{A}| = |C|$.

Reformulating Dilworth's Theorem

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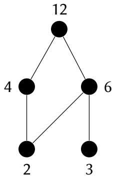

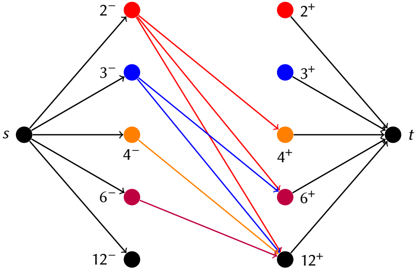

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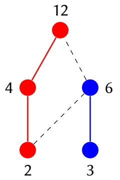
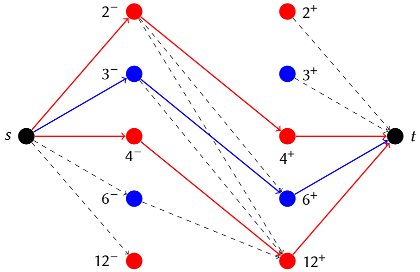
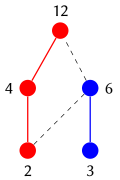
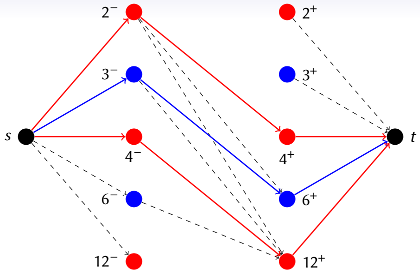
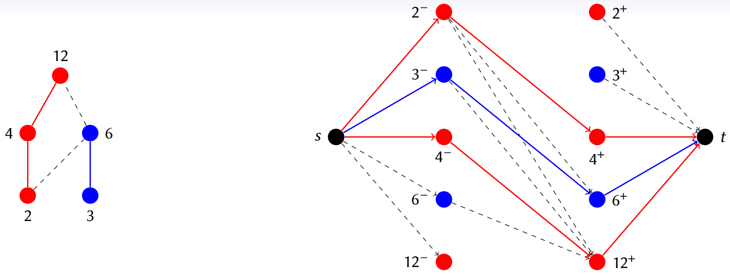








- Let $|f|$ denote the maximum flow of the flow network constructed by the Hasse diagram.



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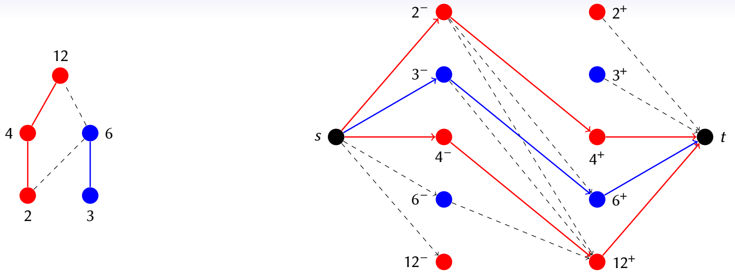

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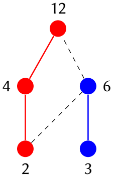
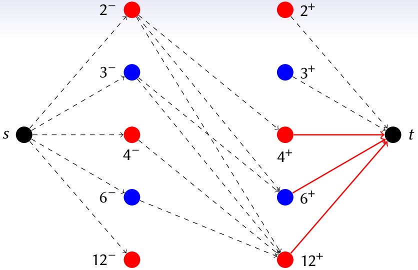
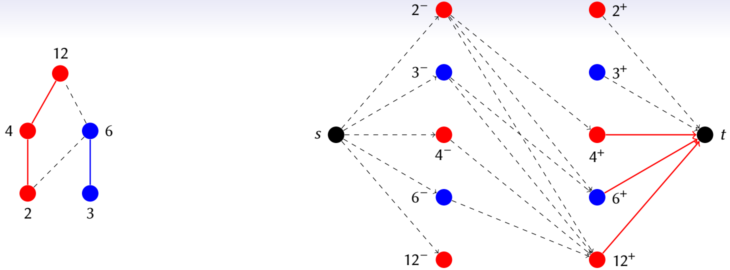
- Then P is partitioned into $|B| = |P| - |f|$ chains.
    - We obtain the two chains in the flow network by following along the paths:

$$\{s \longrightarrow 2^- \longrightarrow 4^+ \longrightarrow 4^- \longrightarrow 12^+ \longrightarrow t\}, \qquad (2 \longrightarrow 4 \longrightarrow 12)$$
$$\{s \longrightarrow 3^- \longrightarrow 6^+ \longrightarrow t\}. \qquad (3 \longrightarrow 6)$$





- We now compute the size of the largest antichain.



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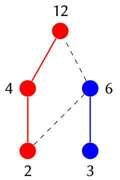

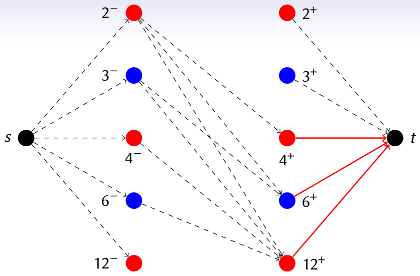

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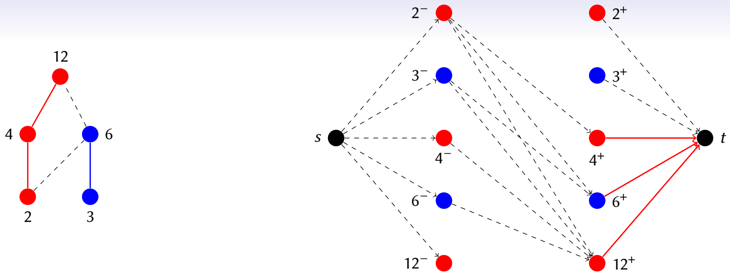

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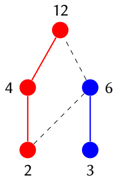
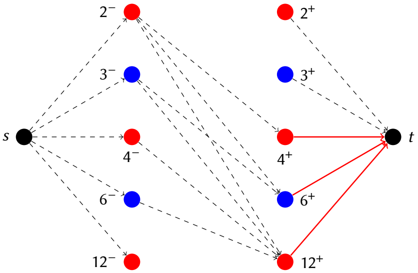
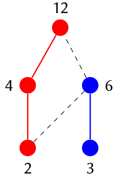
- The only edges that contribute towards the capacity cut are the edges (s, a^-) and (a^+, t). Therefore, this excludes all of the elements in A; that is,

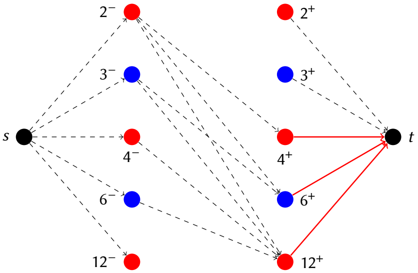
$$c(S, T) = |P| - |A| \implies |A| = |P| - c(S, T).$$





- **Maximum Flow**: $|P| - |B|$ number of partitions. So $|B|$ is minimised (i.e. minimum number of chains).





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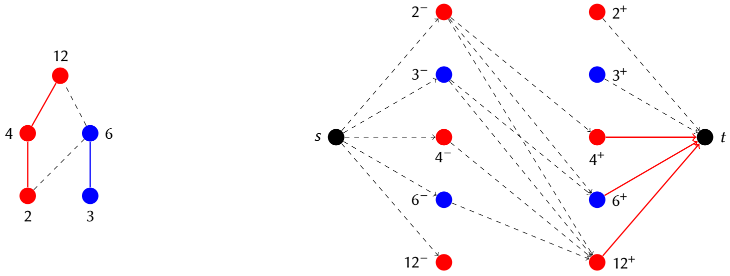

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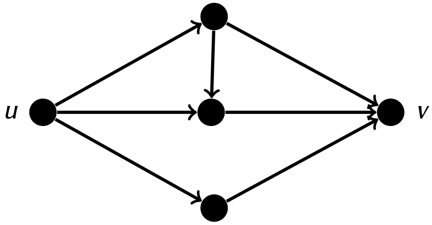
- Therefore, the largest sized antichain corresponds to the smallest number of chains that partition P.

Menger's Theorem

In this problem, we are given a directed and unweighted graph $G = (V, E)$ where $u, v \in V$ are two non-adjacent vertices.

- **Question**: How many edge-disjoint paths are there from u to v?



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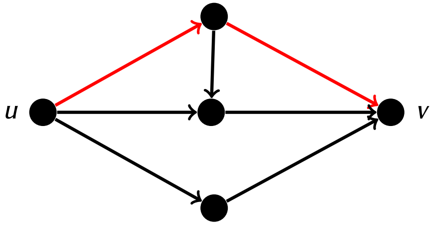

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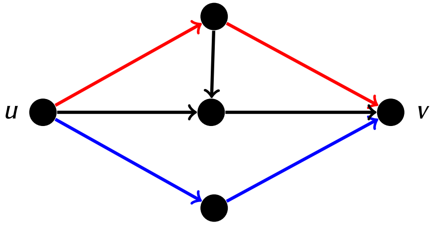

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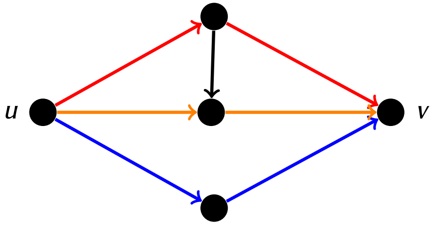


Menger's Theorem

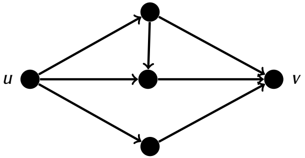
It turns out that the maximum number of edge-disjoint paths from u to v corresponds to the minimum number of edges required to separate u and v!

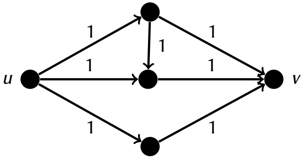
Menger's Theorem

If $u, v \in V$, then there is a (u, v)-separating set of edges S and a collection of edge-disjoint paths $\mathcal{P}$ from u to v such that $|S| = |\mathcal{P}|$.

Reformulating Menger's Theorem

- u is the source and v is the sink vertex.
- Each edge has capacity 1.





- Note that no two $u - v$ paths can share an edge.
  - Therefore, the maximum flow corresponds to the maximum number of edge-disjoint paths from u to v.
- Since each edge has capacity 1, a cut counts the number of edges that pass through the cut.
  - Therefore, the minimum cut corresponds to the minimum number of edges to remove from the graph.

Concluding Remarks

Other theorems that have relations to maximum flow.

- *König's Theorem* – maximal matching.
- *Mirsky's Theorem* – dual of Dilworth's Theorem.
- *Greene's Theorem* – Generalisation of Dilworth's Theorem.