MATH1131/1141 Revision

Algebra

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Chapter 1: Introduction to vectors

Algebraic interpretation of vectors

- Addition of vectors by component:
  - Add each component together and express them in vector form.
  - e.g. $\binom{1}{2} + \binom{3}{4} = \binom{4}{6}$
  - e.g. $(1\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) + (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) = 4\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$
- Subtraction of vectors by component:
  - Subtract the components individually and then express them in vector form.
  - e.g. $\binom{3}{4} - \binom{1}{2} = \binom{2}{2}$
  - e.g. $(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) - (1\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

Chapter 1: Introduction to vectors

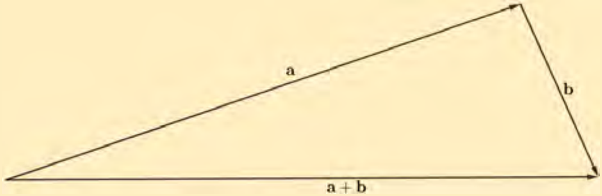
Algebraic interpretation of vectors

- Multiplication of a vector by a non zero scalar:
  - The non zero scalar can be expanded into each of the components.
  - e.g. $-2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$
  - We can also factor out the highest common factor.
  - e.g. $\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
  - We'll develop these ideas a little bit further when we start looking at lines and planes.

Chapter 1: Introduction to vectors

Geometric interpretation of vectors

- Geometrically, a vector is a ray that only requires a **direction** and a **magnitude**.
- Use tail-to-tip method to geometrically add and subtract two vectors.
  - Take a vector **a** and trace the vector from the origin.
  - From the tip (where the arrow would be), trace your second vector **b**. The result from the origin is $\mathbf{a} + \mathbf{b}$.



Chapter 1: Introduction to vectors

Lines

- Lines can be represented in two ways: parametrically and with Cartesian coordinates.
- **Parametrically**: start with a point on the line $\mathbf{a}$ and then walk some distance in the direction of the line:

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \quad \lambda \in \mathbb{R}\}$$

- **Cartesian coordinates**: these appear in the form:

$$S = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \frac{x_1 - a_1}{b_1} = \frac{x_2 - a_2}{b_2} = \cdots = \frac{x_n - a_n}{b_n}\right\}$$

Chapter 1: Introduction to vectors

Lines in parametric form

- We saw the parametric representation of a typical line passing through a point and parallel to a vector.
- We can also express a line segment $\overrightarrow{AB}$ parametrically as:

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda\left(\mathbf{b} - \mathbf{a}\right), \quad \lambda \in [0,1]\}$$

A note about parametric lines!

If you get a different parametric representation, don't worry! You could still be correct. Lines have an infinite number of parametric representations. For example, these two describe the same line. See if you can find the equation of the line!

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Chapter 1: Introduction to vectors

Lines in parametric form - example

MATH1131/1141 2016 Semester 1 Q2vi a)
The points A and B have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}.$$

a) Find a parametric vector equation of the line ℓ passing through A and B.

Chapter 1: Introduction to vectors

We note that the parametric representation of a line is given by:

$$\ell = \mathbf{a} + \lambda \mathbf{v}, \quad \lambda \in \mathbb{R}.$$

For convenience, we'll just choose our **a** vector to be the **a** in the question. But it's perfectly fine to pick **b**.
Our **v** is just the line segment **AB**. This will give us:

$$\begin{aligned} \ell &= \mathbf{a} + \lambda \left(\mathbf{b} - \mathbf{a} \right), \quad \lambda \in \mathbb{R} \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \left(\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}. \end{aligned}$$

Chapter 1: Introduction to vectors

Lines in Cartesian form

The Cartesian form could appear as the following:

$$\frac{x-a}{b} = \frac{y-c}{d}.$$

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part one

For this part, we consider the standard Cartesian equation:

$$ax + by = c$$

To convert to parametric, we:

- Set one variable to be the parameter λ.
- Rewrite the other variable in terms of the parameter.
- Express your answer in vector form.

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part one - example

Consider the Cartesian equation of a line:

$$\ell : 2x - 4y = 6.$$

Find a parametric equation of the Cartesian equation ℓ.

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part one - example

Consider the Cartesian equation of a line:

$$\ell : 2x - 4y = 6.$$

Find a parametric equation of the Cartesian equation ℓ.

Let's set y to be our parameter λ.

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part one - example

Consider the Cartesian equation of a line:

$$\ell : 2x - 4y = 6.$$

Find a parametric equation of the Cartesian equation ℓ.

Let's set y to be our parameter λ. Then we have $2x - 4\lambda = 6$.

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part one - example

Consider the Cartesian equation of a line:

$$\ell : 2x - 4y = 6.$$

Find a parametric equation of the Cartesian equation ℓ.

Let's set y to be our parameter λ. Then we have $2x - 4\lambda = 6$. We then aim to write x in terms of our parameter.

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part one - example

Consider the Cartesian equation of a line:

$$\ell : 2x - 4y = 6.$$

Find a parametric equation of the Cartesian equation ℓ.

Let's set y to be our parameter λ. Then we have $2x - 4\lambda = 6$. We then aim to write x in terms of our parameter. This gives us:

$$x = \frac{6 + 4\lambda}{2} = 3 + 2\lambda, \quad \lambda \in \mathbb{R}.$$

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Lines - From Cartesian to parametric - part one - example

Consider the Cartesian equation of a line:

$$\ell : 2x - 4y = 6.$$

Find a parametric equation of the Cartesian equation ℓ.

Let's set y to be our parameter λ. Then we have $2x - 4\lambda = 6$. We then aim to write x in terms of our parameter. This gives us:

$$x = \frac{6 + 4\lambda}{2} = 3 + 2\lambda, \quad \lambda \in \mathbb{R}.$$

So, our parametric representation COULD be:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 + 2\lambda \\ 0 + \lambda \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part two - example

Modified version of class test 1 from 2014 Semester 1 Version 2b

Consider the line ℓ in $\mathbb{R}^3$ with Cartesian equation:

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}.$$

Find a parametric equation of the line ℓ.

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part two - example

Modified version of class test 1 from 2014 Semester 1 Version 2b

Consider the line ℓ in $\mathbb{R}^3$ with Cartesian equation:

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}.$$

Find a parametric equation of the line ℓ.

Setting the entire equation equal to our parameter λ, we get:

$$\lambda = \frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}.$$

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part two - example

Modified version of class test 1 from 2014 Semester 1 Version 2b

Consider the line ℓ in $\mathbb{R}^3$ with Cartesian equation:

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}.$$

Find a parametric equation of the line ℓ.

Let's solve for x, y and z. If $\lambda = \frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}$, then:

$$\lambda = \frac{x-2}{3} \Rightarrow x = 3\lambda + 2$$

$$\lambda = \frac{y+1}{4} \Rightarrow y = 4\lambda - 1$$

$$\lambda = \frac{z+3}{1} \Rightarrow z = \lambda - 3$$

Chapter 1: Introduction to vectors

Lines - From Cartesian to parametric - part two - example

Modified version of class test 1 from 2014 Semester 1 Version 2b

Consider the line ℓ in $\mathbb{R}^3$ with Cartesian equation:

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}.$$

Find a parametric equation of the line ℓ.

So our parametric representation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+3\lambda \\ -1+4\lambda \\ -3+\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

Lines - From parametric to Cartesian

Here, we'll consider a simple vector in $\mathbb{R}^3$. But this method works for any dimension. We consider the equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

To convert to Cartesian, we:

- write each of the components separately and solve for the parameter λ.
- equate all of the equations to form the alternative form of a line.

Chapter 1: Introduction to vectors

Lines - From parametric to Cartesian

So this will give us:

$$x = x_0 + \lambda x_1 \Rightarrow \lambda = \frac{x - x_0}{x_1}$$
$$y = y_0 + \lambda y_1 \Rightarrow \lambda = \frac{y - y_0}{y_1}$$
$$z = z_0 + \lambda z_1 \Rightarrow \lambda = \frac{z - z_0}{z_1}$$

Finally, equating all of the equations give us the alternatively form of a line:

$$\frac{x - x_0}{x_1} = \frac{y - y_0}{y_1} = \frac{z - z_0}{z_1}.$$

Chapter 1: Introduction to vectors

Lines - From parametric to Cartesian - example

Find the Cartesian form for the line:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

Lines - From parametric to Cartesian - example

Find the Cartesian form for the line:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Writing each component separately (and solving for the parameter) gives us:

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Lines - From parametric to Cartesian - example

Find the Cartesian form for the line:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Writing each component separately (and solving for the parameter) gives us:

$$x = 2 + 3\lambda \Rightarrow \lambda = \frac{x-2}{3}$$

Chapter 1: Introduction to vectors

Lines - From parametric to Cartesian - example

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Lines - From parametric to Cartesian - example

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$$x = 2 + 3\lambda \Rightarrow \lambda = \frac{x-2}{3}$$

$$y = -3 + 5\lambda \Rightarrow \lambda = \frac{y+3}{5}$$

$$z = 1 + 6\lambda \Rightarrow \lambda = \frac{z-1}{6}$$

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Lines - From parametric to Cartesian - example

Find the Cartesian form for the line:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Let's eliminate the parameter! λ is the same in all of the equations.

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Lines - From parametric to Cartesian - example

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Let's eliminate the parameter! λ is the same in all of the equations. So:

$$\lambda = \frac{x-2}{3} = \frac{y+3}{5} = \frac{z-1}{6}.$$

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Lines - From parametric to Cartesian - example

Find the Cartesian form for the line:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Let's eliminate the parameter! λ is the same in all of the equations. So:

$$\lambda = \frac{x-2}{3} = \frac{y+3}{5} = \frac{z-1}{6}.$$

..Voila! Our Cartesian equation!

Chapter 1: Introduction to vectors

Introduction to Planes

- Like lines, planes can also be represented parametrically and with Cartesian coordinates.
- **Parametrically**: start with a point $\mathbf{a}$ and then start walking some distance in two directions:

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda \mathbf{v}_1 + \mu \mathbf{v}_2, \quad \lambda, \mu \in \mathbb{R}\}$$

- **Cartesian**: we represent any plane as:

$$ax + by + cz = d.$$

Chapter 1: Introduction to vectors

Planes in parametric form - example

MATH1131 2015 Semester 2 Q3iv

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$$

Chapter 1: Introduction to vectors

Planes in parametric form - example

MATH1131 2015 Semester 2 Q3iv

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$$

Let's begin by walking up to a point. We'll walk up to the point $(1, 2, -1)^T$.

Chapter 1: Introduction to vectors

Planes in parametric form - example

MATH1131 2015 Semester 2 Q3iv

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$$

Let's begin by walking up to a point. We'll walk up to the point $(1, 2, -1)^T$.

Now, from where we're standing, we want to be able to walk to the second and third points.

Chapter 1: Introduction to vectors

Planes in parametric form - example

MATH1131 2015 Semester 2 Q3iv

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$$

To get to the second point, we can find the direction we want to walk in by taking the first vector from the second vector:

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Chapter 1: Introduction to vectors

Planes in parametric form - example

MATH1131 2015 Semester 2 Q3iv

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$$

Similarly, to get to the third point, we can find the direction we want to walk in by taking the first vector from the third vector:

$$\begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -4 \end{pmatrix}.$$

Chapter 1: Introduction to vectors

Planes in parametric form - example

MATH1131 2015 Semester 2 Q3iv

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$$

Then the parametric form becomes:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

Planes - From Cartesian to parametric

Say that we have a plane of the form:

$$ax + by + cz = d.$$

1. We find three points on the plane.
2. From these points that we find, we set one point as a pivot point.
3. We then find the direction vectors to get our parametric vector form of a plane.

Chapter 1: Introduction to vectors

Planes - From Cartesian to parametric

Say that we have a plane of the form:

$$ax + by + cz = d.$$

Alternatively, we can:

1. Write one variable in terms of the other. You will then have two degrees of freedom.
2. Then we can find the parametric equation of the vector form quite easily!

We shall do an example to demonstrate these two different methods.

Chapter 1: Introduction to vectors

Planes - From Cartesian to parametric – example (Method 1)

Find a parametric vector form of the plane

$$4x_1 - 3x_2 + 6x_3 = 12.$$

Chapter 1: Introduction to vectors

For reference:
We shall begin by finding just three points on the plane. The easiest points are at $(0, 0, x_3)$, $(0, x_2, 0)$ and $(x_1, 0, 0)$. So the three points are:

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

Fixing the first vector to be our point of reference, we can find the direction vectors by subtracting the other vectors from the first vector. So the direction vectors become:

$$\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

Chapter 1: Introduction to vectors

For reference:
Thus, we can express the Cartesian equation as a parametric vector form:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

Planes - From Cartesian to parametric – example (Method 2)

Find a parametric vector form of the plane

$$4x_1 - 3x_2 + 6x_3 = 12.$$

Chapter 1: Introduction to vectors

For reference:
Alternatively, we can rewrite x_1 in terms of x_2 and x_3 by solving for x_1 in terms of x_2 and x_3. Doing so gives us

$$x_1 = \frac{1}{4}(12 + 3x_2 - 6x_3).$$

Then, our parametric equation becomes:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(12 + 3x_2 - 6x_3) \\ x_2 \\ x_3 \end{pmatrix}.$$

This boils down to the parametric equation:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{6}{4} \\ 0 \\ 1 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

Planes - From parametric to Cartesian

To convert from parametric to Cartesian:

1. We write each component separately (like we did for lines).
2. Eliminate the parameters with our equations.

Chapter 1: Introduction to vectors

Planes - From parametric to Cartesian - example

Find the Cartesian form of the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Chapter 1: Introduction to vectors

For reference:

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then comparing each component separately, we have the equations:

$$x_1 = 1 + \lambda - \mu, \quad x_2 = 1 + \lambda, \quad x_3 = 2 + \lambda + 2\mu.$$

By inspection, we deduce that

$$x_1 = x_2 - \mu \Rightarrow \mu = x_2 - x_1.$$

Substituting this equation into the third equation gives us:

$$x_3 = 2 + (x_2 - 1) + 2(x_2 - x_1) = 3x_2 - 2x_1 + 1$$

or:

$$2x_1 - 3x_2 + x_3 = 1.$$

Chapter 1: Introduction to vectors

Applications - lines and planes

MATH1131/1141 2018 Semester 1 Q2vi

Consider the lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$ defined below.

$$\ell_1 : \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

$$\ell_2 : x_1 = 4, \frac{x_2 - 4}{2} = \frac{x_3 + 1}{3}.$$

Show that the lines ℓ_1 and ℓ_2 intersect.

Chapter 1: Introduction to vectors

For reference:
We begin by converting ℓ_2 to parametric form. Setting

$$\lambda = \frac{x_2 - 4}{2} = \frac{x_3 + 1}{3},$$

we get the parametric equation:

$$\ell_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

By inspection, we see that setting $\lambda = 2$ in ℓ_1 will give us the point $(4, 4, -1)$. Hence, they intersect at $(4, 4, -1)$.

Table of Contents

Chapter 2: Vector Geometry

Lengths

- We define the length of a vector **a** to be

$$|\mathbf{a}| = \sqrt{a_1^2 + \ldots + a_n^2}$$

where $a_1^2 + \ldots + a_n^2$ is indicative of the vector components

Chapter 2: Vector Geometry

Dot Product

- The dot product is a form of **scalar multiplication** on vector components that will always yield a real number
- $$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
- We know that if the dot product is 0, then for:

  $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta,$$

  we have $\theta = \frac{\pi}{2}$
- Essentially this means given 2 vectors, we can determine whether or not they are *orthogonal*

Chapter 2: Vector Geometry

The dot product - Arithmetic Properties

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- **Commutative Law:** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$
- **Distributive Law:** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c}$

Chapter 2: Vector Geometry

Projections

- Projections are magnitude of the vector components with respect to another vector
- We define the projection of some vector $\mathbf{a}$ on some other vector $\mathbf{b}$ by the following:

$$\mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

Chapter 2: Vector Geometry

Projections

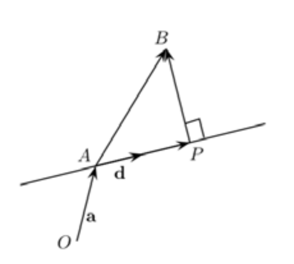
Projections are essentially the heart of Vector Geometry. Problem solving questions in this topic will require you to think critically about how vectors are related to each other through this concept

Chapter 2: Vector Geometry

Projections - An important result

Let's look at a more theoretical question that proves a very common (and important) result

- Given a point B and the line $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$, calculate the shortest distance between the point and the line.



Chapter 2: Vector Geometry

Projections - An important result

- Looking at our figure. We already know the altitude $\overrightarrow{PB} = \overrightarrow{AB} - \overrightarrow{AP}$. We are only interested in the *distance or magnitude* of this vector i.e. $|\overrightarrow{PB}|$.
- Here we have to recognise that $\overrightarrow{AB}$ is nothing but $\overrightarrow{PB} = \mathbf{b} - \mathbf{a} - \overrightarrow{AP}$
- But we know that $\overrightarrow{AP}$ is simply the projection of $\overrightarrow{AB}$ onto $\mathbf{d}$. I.e.

$$\text{proj}_d(\mathbf{b} - \mathbf{a})$$

- Hence we can express our final result, that is the shortest distance, by the following expression

$$|\overrightarrow{PB}| = |\mathbf{b} - \mathbf{a} - \text{proj}_d(\mathbf{b} - \mathbf{a})|$$

Chapter 2: Vector Geometry

Cross Product - Definition

- Sometimes we may want to find a vector that is not perpendicular to 1 but 2 vectors.
- The cross-product is an arithmetic operation on 2 vectors that essentially does this

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Chapter 2: Vector Geometry

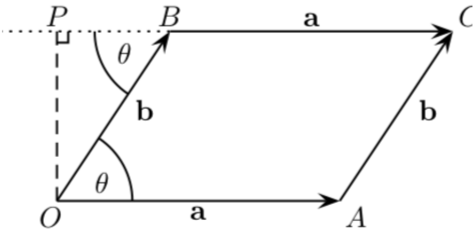
Cross Product - Arithmetic Properties

- $\mathbf{a} \times \mathbf{a} = 0$, i.e., the cross product of a vector with itself is the zero vector.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. The cross product is not **commutative**. If the order of vectors in the cross product is reversed, then the sign of the product is also reversed.
- $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{ab})$ and $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$
- $\mathbf{a} \times (\lambda \mathbf{a}) = 0$, i.e., the cross product of parallel vectors is zero
- Distributive Laws i.e $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{a}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{ac}$.

Chapter 2: Vector Geometry

Cross Product - Areas

The parallelogram formed from 2 vectors has an area equivalent to the magnitude (or length) of the vector yielded from applying the cross product on the two vectors.



Chapter 2: Vector Geometry

Cross Product - Volumes

We can extend this understanding of areas in parallelograms to find volumes of parallelepipeds. Essentially you can think of these solids as prisms, with the base of a parallelogram.

- Like any prism, we know that the volume can be derived simply by multiplying the base by the perpendicular height
- If we write our perpendicular height as a projection of another vector, then we can derive the expression for the volume of a parallelopiped as the following:

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Chapter 2: Vector Geometry

Planes - Point Normal Form

Let's revisit planes, however this time we are going to re-define them with our understanding of vector geometry

- We know that the dot product of the normal vector to a plane and a vector parallel to the plane will always be 0, since the two vectors are perpendicular to each other.
- We can express a normal vector to a plane as

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

Chapter 2: Vector Geometry

Planes - Point Normal Form

- Likewise a vector parallel to the plane can be written as:

$$\mathbf{x} - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

  Where c_1, c_2, c_2 are the vector components of a coordinate vector on the plane

- Thus it follows that our point normal form is expressed as:

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \left(\mathbf{x} - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) = 0$$

Chapter 2: Vector Geometry

Planes - Converting between forms

Point-normal to Cartesian

- Expanding on the definition

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} . \left(\mathbf{x} - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) = 0$$

Gives

$$\begin{aligned} n_1x_1 + n_2x_2 + n_3x_3 &= n_1c1 + n_2c_2 + n_3c_3 \\ &= b \end{aligned}$$

Chapter 2: Vector Geometry

Planes - Converting between forms

Cartesian to Point-normal

- $n_1x_1 + n_2x_2 + n_3x_3 = b$
- On observation we note, that the coefficients of x terms, correspond to the vector components of the normal.
- To obtain our coordinate vector, i.e. **c** we need to find a point that satisfies the plane.
  We can do this by setting fixed values for any 2 variables (0) in our Cartesian equation and solving for the third.

Chapter 2: Vector Geometry

Planes - Converting between forms

Parametric to Point-normal

- Our 2 components for the point-normal form are as the name suggests, a point/coordinate and a normal vector
- In parametric form we have,

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{u} + \lambda_2 \mathbf{v}$$

- So we know to obtain a normal vector we find the Cross product of $\mathbf{u}$ and $\mathbf{v}$ i.e. $\mathbf{u} \times \mathbf{v}$
- Our coordinate vector however can just be taken as $\mathbf{a}$
- We have a **Point** and a **normal**, thus we plug that in to get our point-normal form.

Chapter 2: Vector Geometry

Planes - Converting between forms

Point-normal to Parametric

- First convert **Point-normal to Cartesian** as shown in previous slides
- then convert **Cartesian to parametric** form as explained in chapter 1

Chapter 2: Vector Geometry

Planes - Converting between forms

Parametric to Cartesian

- Previously the only method of conversion was through expanding the components to gain a system of linear equations, and eliminating the parametric
- An easier way is to **convert the parametric vector form into point-normal form**.
- Then we simplify **convert from point-normal to Cartesian**
- It may seem longer because this is a two step method, however the algebra is much simpler.

Chapter 3: Complex numbers

MATH 1131 NOVEMBER 2010 Q2 (vii)

Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are distinct non-zero vectors with the property that

$$\text{proj}_{\mathbf{w}}(\mathbf{u}) = \text{proj}_{\mathbf{w}}(\mathbf{v})$$

Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{w}$

Chapter 3: Complex numbers

Condition: $\text{proj}_w(\mathbf{u}) = \text{proj}_w(\mathbf{v})$

1. Recall ,

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

2. Thus,

$$\frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$

3. Simplifying we have

$$\mathbf{u}\cdot\mathbf{w} - \mathbf{v}\cdot\mathbf{w} = 0$$

$$\mathbf{w}\cdot(\mathbf{u}-\mathbf{v}) = 0$$

4. By definition of dot product $\mathbf{u}-\mathbf{v}$ is perpendicular to $\mathbf{w}$

Chapter 3: Complex numbers

MATH 1141 JUNE 2012 Q2 (iii)

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$

Chapter 3: Complex numbers

We don't really know where to start, so lets pretend we know the answer and work backwards until we get to a point we are familiar with.

If they are perpendicular then

$$
\begin{aligned}
&(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = 0\\
&(\mathbf{u}\cdot\mathbf{u}) + (\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}) - (\mathbf{v}\cdot\mathbf{v}) = 0\\
&\mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = 0\\
&|\mathbf{u}|^2 - |\mathbf{v}|^2 = 0\\
&|\mathbf{u}|^2 - |\mathbf{u}|^2 = 0
\end{aligned}
$$

In our actual proof, we would write it backwards.

Chapter 3: Complex numbers

Tip

- We learnt a useful strategy when we can't immediately formulate how to answer the question.
- Instead of answering the question traditionally. Why not assume you have answered the question, and figure out how you would have got to that solution, by working backwards.
- This is called a 'Discovery', essentially playing around with the answer until it looks familiar.

Table of Contents

Chapter 3: Complex numbers

A review of the system

Just like with our real numbers, we can add, subtract, multiply and divide complex numbers.

- $(x + iy) \pm (a + ib) = (x \pm a) + i(y \pm b)$
- $(x + iy)(a + ib) = (ax - by) + i(bx + ay)$ Distributive Law

Chapter 3: Complex numbers

Dividing Complex Numbers

The conjugate of a complex number $z = a + ib$ is given by $\bar{z} = a - ib$. We flip the imaginary part.

$$\frac{x+iy}{a+ib} = \frac{x+iy}{a+ib} \times \frac{a-ib}{a-ib}$$

- Note the technique when dividing complex numbers is to multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(x+iy)(a-ib)}{a^2 - b^2}$$

We have **realised** the denominator for further manipulation

Chapter 3: Complex numbers

Properties of Polar Form - Modulus

The Cartesian form only gives information about the coordinates of a complex number. **Polar form** however,

- Considers the complex number as a **vector**
- An associated **modulus -** r and **argument -** θ
- We define the polar form as: $r(\cos\theta + i\sin\theta)$
- Hence it follows that $x = r\cos\theta$ and $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$. Allows conversion between the 2 forms
- We can also define $re^{i\theta}$
- Polar form allows us to easily manipulate vectors through their **geometric properties**

Chapter 3: Complex numbers

Properties of Polar Form

Conjugate in Polar Form

- The conjugate in Polar Form is given by

$$\begin{aligned} z &= r\cos\theta - r\sin\theta \\ &= r\cos(-\theta) + r\sin(-\theta) \\ &= re^{-i\theta} \end{aligned}$$

- Geometrically this means that the conjugate of a complex number in Polar form has a **negative argument** and thus is **reflected across the Real axis**

Chapter 3: Complex numbers

Properties of Polar Form

Modulus Properties

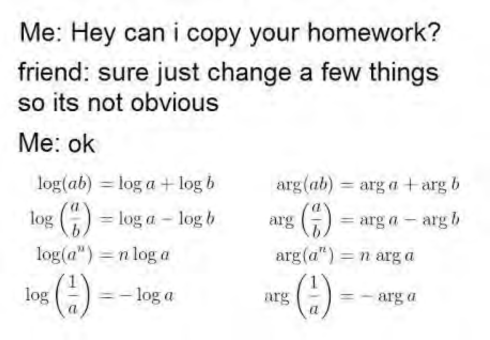
- $|a||b| = |ab|$
- $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$
- $|z|^n = |z^n|$
- $z\bar{z} = |z|^2$
- $\bar{z} = \frac{1}{z}$ iff $|z| = 1$

Chapter 3: Complex numbers

Properties of Polar Form

Argument Properties

- $\arg(ab) = \arg(a) + \arg(b)$
- $\arg\left(\frac{a}{b}\right) = \arg a - \arg b$
- $\arg(z)^n = n\arg(z)$



Chapter 3: Complex numbers

Properties of Polar Form

De Moivre's Theorem

The above properties are suffice for use to multiply and divide complex numbers in polar form.

However, polar form also allows simple manipulation of powers of complex numbers through De Moivre's Theorem

- $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

Chapter 3: Complex numbers

Roots of Unity

With the aid of De Moivre's theorem we can know plot the roots of some basic complex numbers. **e.g.** $z^n = 1$

- Let $z = re^{i\theta}$ and convert 1 into a polar vector i.e. e^{i0}
- We have: $r^n e^{i(n\theta)} = e^{0i}$ (By De Moivre's Theorem)
- Equating Modulus and Argument gives, $r = 1$ and $\theta = \frac{0+2\pi k}{n}$. Since we can add any multiple of 2π
- Our n roots of unity are

$$z = e^{i\left(\frac{0+2\pi k}{n}\right)}$$

- For distinct roots, we choose consecutive values of k such that θ lies in $(-\pi, \pi]$

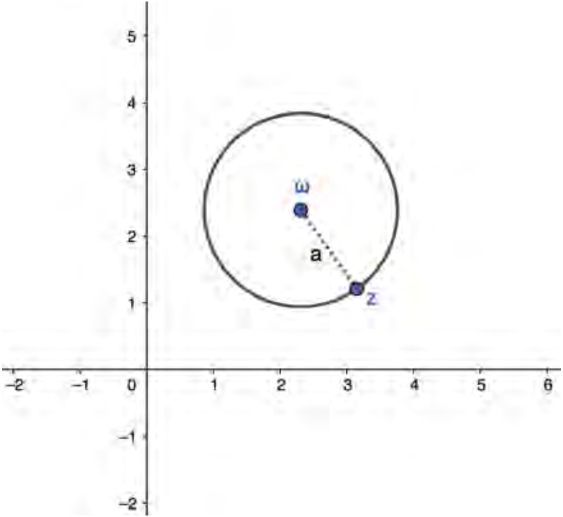
Chapter 3: Complex numbers

Locus and Regions - Modulus

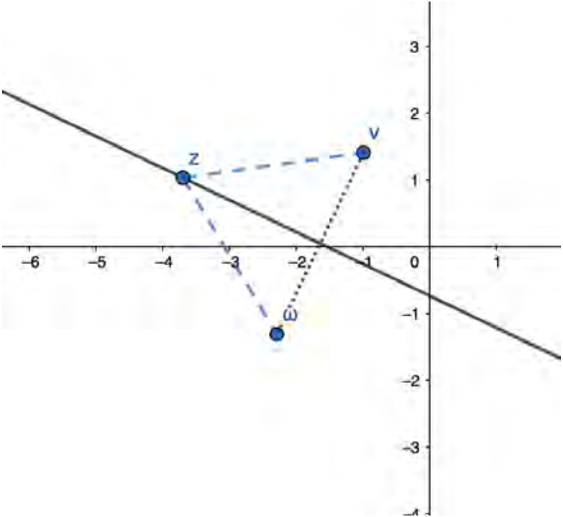
.

- $|z - \omega| = a$. Circle with origin at ω and radius of a
- $|z - \omega| = |z - v|$. Line of perpendicular bisector between the points of ω and v
- If we had an inequality, then these loci would become regions instead
- We would consider the region inside or outside of the circle for the first point
- And we would consider the region on either side of the perpendicular bisector for the second point

Chapter 3: Complex numbers



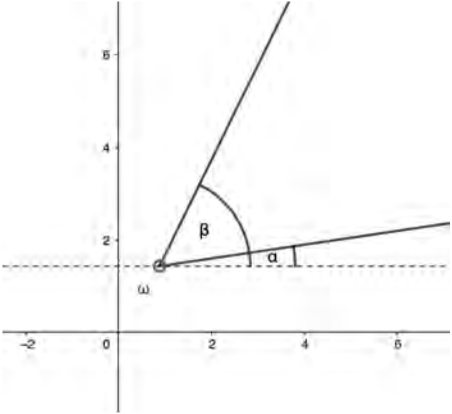
Chapter 3: Complex numbers



Chapter 3: Complex numbers

Locus and Regions - Argument

- $\alpha \leq \arg(z - \omega) \leq \beta$



Chapter 3: Complex numbers

Locus and Regions - Re(z) and Im(z)

- $\text{Re}(z) = a$ Corresponds to the line $x = a$
- $\text{Im}(z) = a$ Corresponds to the line $y = a$
- As discussed before, these become regions if we replace the equality with an inequality

Chapter 3: Complex numbers

Trig Applications

Converting $\cos(n\theta)$ **into terms of** $\cos\theta$

- Express $(\cos\theta + i\sin\theta)^n$ as $\cos(n\theta) + i\sin(n\theta)$
- Express $(\cos\theta + i\sin\theta)^n$ using the binomial theorem and combine real and imaginary parts
- Equate expressions 1 and 2.
- For an expression of $\cos(n\theta)$, equate **Real** parts and for an expression of $\sin(n\theta)$, equate **Imaginary** parts

Chapter 3: Complex numbers

Trig Applications

Q) Convert $\cos(3\theta)$ into terms of $\cos\theta$

1. Expand $(\cos\theta + i\sin\theta)^3$ one way using **DeMoivre's** and another way with **binomial theorem**
2. $\cos(3\theta) + i\sin(3\theta)$
3. $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$
4. Comparing Real parts from (2) and (3)

$$\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Chapter 3: Complex numbers

Trig Applications

Converting $\cos^n(\theta)$ into linear terms

Proposition: $z^j + \frac{1}{z^j} = 2\cos(j\theta)$ (since $|z| = 1$)

- Let $z = \cos\theta + i\sin\theta$
- We know $\cos^n\theta = \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^n$
- Expand the RHS using the binomial theorem
- Group terms in the form of $z^j + \frac{1}{z^j}$ and write them as $2\cos(j\theta)$. (Using our Proposition)
- Similarly $\sin^n\theta = \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^n$

Chapter 3: Complex numbers

Trig Applications

Q) Convert $\cos^3\theta$ into linear terms

1. Let $z = \cos\theta + i\sin\theta$
2. We know,

$$\begin{aligned}\cos^3\theta &= \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^3 \\ &= \frac{1}{8}\left(\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)\right) \\ &= \frac{1}{8}\left(2\cos(3\theta) + 6\cos(3\theta)\right)\end{aligned}$$

Chapter 3: Complex numbers

Complex Polynomials

Factorisation Theorem

- Every polynomial of $p(z)$ of degree $n \geq 1$ can be factorised into linear factors of the form

$$p(z) = a(z - \alpha_1)...(z - \alpha_n)$$

Complex Conjugate Theorem

- if $(z - \alpha)$ is a factor of $p(z)$ then so is $(z - \bar{\alpha})$

Chapter 3: Complex numbers

Complex Polynomials

Factorising into Quadratic Factors with real coefficients

- $(z-\omega)(z-\bar{\omega}) = z^2 - 2\,\mathrm{Re}(\omega)z + |\omega|^2$

Note **none of the coefficients are complex numbers**!!!. This identity is very important as it enables us to convert complex linear factors into quadratic factors with only real coefficients

Chapter 3: Complex numbers

MATH 1141 JUNE 2015 Q4 (iii)

a) Use De Moivre's theorem to express $\sin 5\theta$ as a polynomial in $x = \sin\theta$

1. Expand $(\cos\theta + i\sin\theta)^5$ using the binomial theorem.
2. Apply De Moivre's on $(\cos\theta + i\sin\theta)^5$
3. Equating imaginary parts in 1 and 2 yields

$$\sin 5\theta = 16\sin^5(x) - 20\sin^3(x) + 5\sin(x)$$

4. Let $x = \sin(\theta)$ therefore

$$\sin(5\theta) = 16x^5 - 20x^3 + 5x$$

where $x = \sin\theta$

Chapter 3: Complex numbers

MATH 1141 JUNE 2015 Q4 (iii)

b) Consider the polynomial $p(x) = 16x^5 - 20x^3 + 5x - 1$. Show that $\sin(\frac{\pi}{10})$ is a root of $p(x)$.

1. Use part a) to simplify the polynomial into a trig expression.
2. We have $p(x) = \sin(5\theta) - 1$ where $x = \sin\theta$
3. Hence the roots occur when $\sin(5\theta) = 1$
4. At $\theta = \frac{\pi}{10}$ the expression equals 1. Hence $\sin\frac{\pi}{10}$ is a root of $p(x)$

Chapter 3: Complex numbers

MATH 1141 JUNE 2015 Q4 (iii)

c) Using the fact

$$16x^5 - 20x^3 + 5x - 1 = (x-1)(4x^2 + 2x - 1)^2$$

find the distinct roots of $p(x)$

1. We know that $x = 1$ is a root, to find the other roots we simply apply the quadratic formula on the second factor.
2. The distinct roots of $p(x)$ are: $z = 1, \frac{-1+\sqrt{5}}{4}, \frac{-1+\sqrt{5}}{4}$

Chapter 3: Complex numbers

MATH 1141 JUNE 2015 Q4 (iii)

d) Evaluate $\sin\frac{\pi}{10}$ in surd form

1. We know $\sin\frac{\pi}{10}$ is the smallest positive root of p(x)
2. By equating the smallest positive root from part c) we conclude that $\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}$

Chapter 3: Complex numbers

MATH 1141 JUNE 2012 Q2 (iii)

Suppose that z lies on the unit circle in the complex plane.
a) Show that $z + \frac{1}{z}$ is real

1. z lies on the unit circle, therefore $|z| = 1$
2. Hence $\bar{z} = \frac{1}{z}$
3. Thus the expression now becomes

$$z + \bar{z}$$

$$2\,\mathrm{Re}(z)$$

4. Which is real. QED

Chapter 3: Complex numbers

MATH 1141 JUNE 2012 Q2 (iii)

Suppose that z lies on the unit circle in the complex plane.
b) Find the maximum value of $z + \frac{1}{z}$

1. Recall, that $|z| = 1$
2. Let $z = \cos\theta + i\sin\theta$
3. Thus
$$z + \bar{z} = 2\,\mathrm{Re}(z) = 2\cos\theta$$
4. The maximum value is 2.

Table of Contents

Chapter 4: System of linear equations

You could already solve a low number of linear equations. For example:

$$x + y = 9$$
$$2x - y = 0$$

Chapter 4: System of linear equations

You could already solve a low number of linear equations. For example:

$$\begin{aligned} x + y &= 9 \\ 2x - y &= 0 \end{aligned}$$

What about:

$$\begin{aligned} x + y + z &= 9 \\ 2x - y + 2z &= 0 \\ 3x + y + 3z &= 10 \end{aligned}$$

Chapter 4: System of linear equations

You could already solve a low number of linear equations. For example:

$$x + y = 9$$
$$2x - y = 0$$

What about:

$$x + y + z = 9$$
$$2x - y + 2z = 0$$
$$3x + y + 3z = 10$$

Still doable? Ok, what about:

$$x + y + z + a = 9$$
$$2x - y + 2z + 2a = 0$$
$$3x + y + 3z - 3a = 10$$
$$4x - y + 4z + 4a = 20$$

Chapter 4: System of linear equations

Revising the augmented matrix

Recall that the augmented matrix look like:

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right)$$

and can be read as:

$$ax + by = e.$$
$$cx + dy = f.$$

Chapter 4: System of linear equations

Elementary row operations

We'll look at four ways of performing elementary row operations:

1. swapping two rows.
2. adding or subtracting two rows together.
3. multiplying a row by a scalar.
4. a combination of the three.

Chapter 4: System of linear equations

Swapping two rows

We can swap two rows. The notation for swapping two rows is:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} d & e & f \\ a & b & c \end{pmatrix}$$

This is very useful for situations where a particular column is non leading as we are trying to reduce it into row echelon. For example:

$$\begin{pmatrix} \boxed{0} & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & \boxed{0} & 1 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Chapter 4: System of linear equations

Adding/subtracting two rows together

We can also add or subtract two rows together. The notation for this is:

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \xrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} 2a & 2b & 2c \\ a & b & c \end{pmatrix}$$

Note that only the first row is affected.

Chapter 4: System of linear equations

Multiplying a row by a scalar

Similarly, we could also multiply a single row by a nonzero scalar. Doing so, we get this notation.

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \xrightarrow{R_1 = 2R_1} \begin{pmatrix} 2a & 2b & 2c \\ d & e & f \end{pmatrix}$$

Also, note that only the first row is affected.

Chapter 4: System of linear equations

Gaussian elimination – reduction to row echelon

1. We perform a combination of elementary row operations onto the matrix.
2. The idea is that we want to write in a form similar to:

$$\begin{pmatrix} \boxed{1} & a & b \\ 0 & \boxed{c} & d \\ 0 & 0 & \boxed{1} \end{pmatrix}.$$

Chapter 4: System of linear equations

Back substitution

1. When we have the matrix in row echelon form, we can just read off the solutions from the last row up.
2. We substitute each solution into the equations above and solve for the rest!

We will do examples of these so don't worry if it's not clicking!

Chapter 4: System of linear equations

Reducing down to row echelon - example

MATH1131 2014 Semester 2 Q4iii

The current ages of Xena, Yenny and Zac are x, y and z years respectively. You are given that 48 years ago Zac's age was triple that of Yenny's age at that time.

a) Explain why $3y - z = 96$.

b) It is also known that the sum of their current ages is 200 and currently Xena's age is the sum of Yenny's and Zac's ages. Set up a system of 3 equations in the three unknowns x, y and z.

c) By reducing your system to echelon form and back substituting, find the current ages of Xena, Yenny and Zac.

Chapter 4: System of linear equations

Reducing down to row echelon - example

MATH1131 2014 Semester 2 Q4iii

The current ages of Xena, Yenny and Zac are x, y and z years respectively. You are given that 48 years ago Zac's age was triple that of Yenny's age at that time.

a) Explain why $3y - z = 96$.

48 years ago, Yenny's age was $y - 48$. But at that time, Zac's age was triple Yenny's age. So,

$$\begin{aligned} z - 48 &= 3(y - 48) \\ z &= 3y - 144 + 48. \\ \therefore 3y - z &= 96. \end{aligned}$$

Chapter 4: System of linear equations

Reducing down to row echelon - example

MATH1131 2014 Semester 2 Q4iii
The current ages of Xena, Yenny and Zac are x, y and z years respectively. You are given that 48 years ago Zac's age was triple that of Yenny's age at that time.

b) It is also known that the sum of their current ages is 200 and currently Xena's age is the sum of Yenny's and Zac's ages. Set up a system of 3 equations in the three unknowns x, y and z.

The set of equations are:

$$3y - z = 96. \tag{1}$$

$$x + y + z = 200. \tag{2}$$

$$x - y - z = 0. \tag{3}$$

Chapter 4: System of linear equations

Reducing down to row echelon - example

MATH1131 2014 Semester 2 Q4iii

The current ages of Xena, Yenny and Zac are x, y and z years respectively. You are given that 48 years ago Zac's age was triple that of Yenny's age at that time.

c) By reducing your system to echelon form and back substituting, find the current ages of Xena, Yenny and Zac.

From part b), we can form an augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 3 & -1 & 96 \\ 1 & 1 & 1 & 200 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

Chapter 4: System of linear equations

Reducing down to row echelon - example

MATH1131 2014 Semester 2 Q4iii

The current ages of Xena, Yenny and Zac are x, y and z years respectively. You are given that 48 years ago Zac's age was triple that of Yenny's age at that time.

c) By reducing your system to echelon form and back substituting, find the current ages of Xena, Yenny and Zac.

In reduced row echelon, we have:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 49 \\ 0 & 0 & 1 & 51 \end{array}\right)$$

$\therefore x = 100, y = 49, z = 51$

Chapter 4: System of linear equations

Deducing the number of solutions

Suppose you have an augmented matrix in row echelon form. Then:

1. there exists no solutions if the right hand side is a leading column.
2. there exists a unique solution if **every left hand column is a leading column** AND the right hand side is not a leading column.
3. there exists infinitely many solutions if there is a column on the left side that's not leading.

Chapter 4: System of linear equations

Deducing the number of solutions - example

MATH1131 2015 Semester 2 Q4iii

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & (\alpha-3)(\alpha-1) & \alpha-1 \end{array}\right)$$

a) For which value of α will the system have no solution?

b) For which value of α will the system have infinitely many solutions?

c) For the value determined in b), find the general solution.

Chapter 4: System of linear equations

Deducing the number of solutions - example

MATH1131 2015 Semester 2 Q4iii

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & (\alpha-3)(\alpha-1) & \alpha-1 \end{array}\right)$$

a) For which value of α will the system have no solution?

If the system has no solutions, then that means the right hand side column is leading column. This implies that $(\alpha-3)(\alpha-1)=0$ and $\alpha-1\neq 0$. So, we deduce that $\alpha=3$ produces a system with no solutions.

Chapter 4: System of linear equations

Deducing the number of solutions - example

MATH1131 2015 Semester 2 Q4iii

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & (\alpha-3)(\alpha-1) & \alpha-1 \end{array}\right)$$

b) For which value of α will the system have infinitely many solutions?

If the system have infinitely many solutions, then a left hand column is not leading and the right hand column is not leading. So, $(\alpha-3)(\alpha-1)=0$ and $\alpha-1=0$. Thus, we get $\alpha=1$.

Chapter 4: System of linear equations

Deducing the number of solutions - example

MATH1131 2015 Semester 2 Q4iii

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & (\alpha-3)(\alpha-1) & \alpha-1 \end{array}\right)$$

c) For the value determined in b), find the general solution.

Row 2 tells us that $3y + 2z = 0 \Rightarrow 3y = -2z$. Row 1 tells us that $x - 2y + z = 4$. Using these two equations, we end up with:

$$2x - 7y = 8.$$

Chapter 4: System of linear equations

Application – span of multiple vectors

We say that a vector V belongs in the span of multiple vectors if there exists a **linear combination** of V using the vectors in the spanning set.

Chapter 4: System of linear equations

Application – span of multiple vectors

For example:

$$\begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$

is in the span of

$$\left\{ \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} \right\}$$

since

$$\begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$$

Chapter 4: System of linear equations

Span of multiple vectors – example

Does $\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$ belong to the span of $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right\}$?

Chapter 4: System of linear equations

For reference:

To see if the vector belong to the span, we need to find a linear combination, namely:

$$\lambda \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}.$$

Placing this into an augmented matrix and row reducing, we get:

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 22 \\ 0 & 0 & 0 \end{array}\right).$$

Since the right hand column is a leading column, we deduce that there are **no** solutions. So the vector **does not** belong in the span.

Table of Contents

Chapter 5: Matrices

Matrix arithmetic

A matrix has a number of columns and rows.

1. $m \times n$ means m rows and n columns. So, a 2×3 matrix means 2 rows and 3 columns.
2. Number of rows = number of columns means the matrix is **square**.
3. Two matrices can be added together if they have the same number of columns AND rows.
4. Matrices are NOT commutative. That is:

$$\boxed{AB \neq BA.}$$

Chapter 5: Matrices

Definition (matrix multiplication)

Let A be an $m \times n$ matrix and X be an $n \times p$ matrix and let $\mathbf{x}_j$ be the jth column of X. Then the product $B = AX$ is the $m \times p$ matrix whose jth column b_j is given by:

$$b_j = AX_j \quad \text{for } 1 \leq j \leq p.$$

Chapter 5: Matrices

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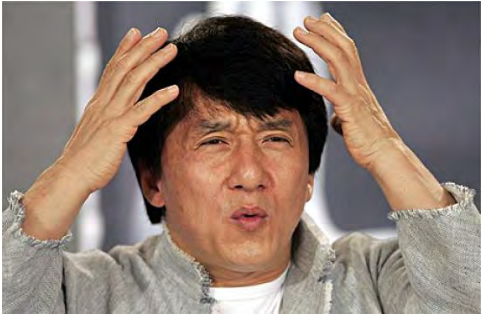

Chapter 5: Matrices

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Let A be an $m \times n$ matrix and X be an $n \times p$ matrix and let $\mathbf{x}_j$ be the jth column of X. Then the product $\mathrm{B} = \mathrm{AX}$ is the $m \times p$ matrix whose jth column b_j is given by:

$$b_j = AX_j \quad \text{for } 1 \leq j \leq p.$$

Alternate definition (Matrix multiplication)

Two matrices can be multiplied together if and only if:

1. The number of columns of the first matrix matches the number of rows of the second matrix.
2. The resulting matrix will be the rows of the first matrix multiplied by the column of the second matrix.

Chapter 5: Matrices

Matrix multiplication - example

MATH1131 2018 Semester 2 Q4iii

Consider the matrices

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & -4 \\ 0 & 10 \\ 1 & -6 \end{pmatrix}.$$

Calculate CD.

Chapter 5: Matrices

For reference:
First, we see that C is a $2 \times \mathbf{3}$ matrix while D is a $\mathbf{3} \times 2$ matrix. Thus, CD is compatible. Next, we observe that CD will be a 2×2 matrix. Performing matrix multiplication, we get:

$$\begin{aligned} CD &= \begin{pmatrix} 1 \times 1 + 1 \times 0 + 1 \times 1 & 1 \times -4 + 1 \times 10 + 1 \times -6 \\ 1 \times 1 + 0 \times 0 + -1 \times 1 & 1 \times -4 + 0 \times 10 + -1 \times -6 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned}$$

Chapter 5: Matrices

Transposes of matrices and properties

Transposes have the notation: A^T.

1. Flip the columns and rows, so that columns become rows and rows become columns. For example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

2. $(AB)^T = B^T A^T$.
3. $(A^T)^T = A$.
4. $(A+B)^T = A^T + B^T$.
5. $A^T = A \implies A$ is symmetric.
6. $A^T = -A \implies A$ is skew-symmetric.

Chapter 5: Matrices

Determinants of matrices and properties

Determinants have the notation: $\det(A) = |A|$. Note that they're **only** defined for square matrices.

1. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant is:

$$\det(A) = ad - bc.$$

Chapter 5: Matrices

Determinants of matrices and properties

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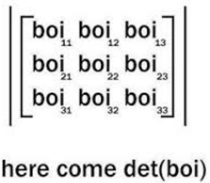


Chapter 5: Matrices

Determinants of matrices and properties

Determinants have the notation: $\det(A) = |A|$. Note that they're **only** defined for square matrices.

1. For a 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

the determinant is:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Be careful about sign changes!

Determinants are tricky because they have sign changes! They follow the pattern: +/-/+/...

Chapter 5: Matrices

Determinants - 3x3 example

MATH1131 2014 Semester 1 Q3ii

Let $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$.

1. Evaluate the determinant of M.

Chapter 5: Matrices

Determinants - 3x3 example

MATH1131 2014 Semester 1 Q3ii

Let $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$.

1. Evaluate the determinant of M.

Determinant comes out to be:

$$\begin{aligned} \det(M) &= 1(1-6) + 1(2+3) + 1(4+1) \\ &= -5 + 5 + 5 \\ &= 5. \end{aligned}$$

Chapter 5: Matrices

Determinants of matrices and properties

Determinants have the notation: $\det(A) = |A|$. Note that they're **only** defined for square matrices.

1. For a 4×4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

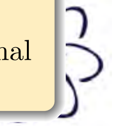
the determinant is: uhh.. let's not go into that.

Chapter 5: Matrices

Using row reduction to find the determinant

For higher dimensions, calculating the determinant by hand becomes ew! So let's find another way to find a determinant. Why don't we try to find the determinant through row reduction? Note that:

1. Multiplying a row by a scalar also scales the determinant. For example, performing the row operation: $R_1 \Rightarrow 2R_1$ scales the determinant by 2.
2. Swapping two rows negates the determinant. For example: $R_1 \Leftrightarrow R_2 \implies -\det(A)$.
3. Adding two rows does not change the determinant.
4. The determinant just becomes the product of the diagonal entries! How easy is that?!?!?



Chapter 5: Matrices

Using row reduction to find the determinant – example

Find the determinant of:

$$\begin{pmatrix} 0 & 3 & 6 \\ 1 & 5 & -2 \\ 1 & 2 & -4 \end{pmatrix}$$

Chapter 5: Matrices

Using row reduction to find the determinant – example

Find the determinant of:

$$\begin{pmatrix} 0 & 3 & 6 \\ 1 & 5 & -2 \\ 1 & 2 & -4 \end{pmatrix}$$

The row operations you will perform may be:

$$R_1 = R_1/3.$$

This will allow us to bring out the three from outside the determinant.

$$R_1 \Leftrightarrow R_2.$$

This will negate the sign of the determinant.

$$R_3 = R_1 - R_3.$$

This will do nothing to the determinant.

Chapter 5: Matrices

Using row reduction to find the determinant – example

Find the determinant of:

$$\begin{pmatrix} 0 & 3 & 6 \\ 1 & 5 & -2 \\ 1 & 2 & -4 \end{pmatrix}$$

When we put all of these together, the determinant becomes -12.

Chapter 5: Matrices

Determinants of matrices and properties

Determinants have the notation: $\det(A) = |A|$. Note that they're **only** defined for square matrices.

1. $\det(A) = \det\left(A^T\right)$.
2. $\det(AB) = \det(A)\det(B)$.

Chapter 5: Matrices

Chapter 5: Matrices

Be careful!

Even though det(AB) = det(A)det(B), we can't say the same about $A + B$. That is, in general,

$$\boxed{\det(A + B) \neq \det(A) + \det(B).}$$

We're going to see an example of where this doesn't hold.

Chapter 5: Matrices

Determinant - example

MATH1131 2018 Semester 2 Q4i

Let A and B be 2×2 matrices.

1. Use a counterexample to show that $\det(A+B)$ does not equal to $\det(A) + \det(B)$ in general.

Chapter 5: Matrices

Determinant - example

MATH1131 2018 Semester 2 Q4i

Let A and B be 2×2 matrices.

1. Use a counterexample to show that $\det(A + B)$ does not equal to $\det(A) + \det(B)$ in general.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Determinant of both is 0.

Chapter 5: Matrices

Determinant - example

MATH1131 2018 Semester 2 Q4i

Let A and B be 2×2 matrices.

1. Use a counterexample to show that $\det(A + B)$ does not equal to $\det(A) + \det(B)$ in general.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Determinant of both is 0.

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } |A + B| = 1.$$

Chapter 5: Matrices

Determinant - another example

MATH1141 2013 Semester 1 Q3iii

Which of the following statements are true **for all** non-zero 2×2 matrices? For those statements which are not always true, give a counter example.

a) $\det(AB) = \det(BA)$.

b) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

c) If $AB = AC$ then $B = C$.

Chapter 5: Matrices

Determinant - another example

MATH1141 2013 Semester 1 Q3iii

Which of the following statements are true **for all** non-zero 2×2 matrices? For those statements which are not always true, give a counter example.

a) $\det(AB) = \det(BA)$.

b) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

c) If $AB = AC$ then $B = C$.

a) Yes. Exploit the property $\det(AB) = \det(A)\det(B)$.

b) No. Counter example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Chapter 5: Matrices

Determinant - another example

MATH1141 2013 Semester 1 Q3iii

Which of the following statements are true **for all** non-zero 2×2 matrices? For those statements which are not always true, give a counter example.

a) $\det(AB) = \det(BA)$.

b) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

c) If $AB = AC$ then $B = C$.

c) No. Counter example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Chapter 5: Matrices

Some more interesting properties of determinants

1. If an entire row or column is 0, then the determinant of the matrix is 0.
2. If a row is a multiple of another row, then the determinant of the matrix is 0.
3. If a column is a multiple of another column, then the determinant of the matrix is 0.
4. Suppose that we have $A = n \times n$ matrix. Then

$$\boxed{\det(mA) = m^n \det(A).}$$

Chapter 5: Matrices

Some more interesting properties of determinants

1. If an entire row or column is 0, then the determinant of the matrix is 0.
2. If a row is a multiple of another row, then the determinant of the matrix is 0.
3. If a column is a multiple of another column, then the determinant of the matrix is 0.
4. Suppose that we have $A = n \times n$ matrix. Then

$$\boxed{\det(mA) = m^n \det(A).}$$

..what hax.

Chapter 5: Matrices

Inverses of matrices and properties

A matrix A has an inverse if and only if:

$$\boxed{\det(A) \neq 0.}$$

Alternatively, we can also say that A is invertible.

1. $AA^{-1} = I$.
2. $\left(A^{-1}\right)^{-1} = A$.
3. $\det\left(A^{-1}\right) = (\det(A))^{-1}$.

Chapter 5: Matrices

Inverse of a 2×2 matrix

The inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Chapter 5: Matrices

Inverse of a 2×2 matrix – example

MATH1131 2014 Semester 1 Q3ii

Let $N = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

Write down the inverse of N.

Chapter 5: Matrices

Inverse of a 2×2 matrix – example

MATH1131 2014 Semester 1 Q3ii

Let $N = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

Write down the inverse of N.

The inverse of a matrix exists if its determinant is not 0. Calculating the determinant gives us 2, so there exists an inverse. Thus, the inverse is:

$$N^{-1} = \frac{1}{\det(N)} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}.$$

Chapter 5: Matrices

Using row reduction to find the inverse

1. Augment the matrix in the form $\left(\begin{array}{c|c} A & I \end{array} \right)$.
2. We row reduce the matrix so that we have $\left(\begin{array}{c|c} I & A^{-1} \end{array} \right)$.
3. The right hand matrix is the inverse of A.

We'll do a few examples, starting with 2×2 matrices.

Chapter 5: Matrices

Using row reduction to find the inverse – 2×2 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

Chapter 5: Matrices

Using row reduction to find the inverse – 2×2 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

Augmenting the matrix, we have:

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right).$$

We aim to have the left side become the identity matrix. Performing elementary row operations, we get:

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1/2 \\ 0 & 1 & -2 & 3/2 \end{array}\right).$$

Chapter 5: Matrices

Using row reduction to find the inverse – 2×2 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

So the inverse of A is the right side matrix:

$$\begin{pmatrix} 1 & -1/2 \\ -2 & 3/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$

as we saw before.

Chapter 5: Matrices

Using row reduction to find the inverse – 3×3 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 8 & 2 & 2 \\ 4 & 8 & 10 \end{pmatrix}$$

Chapter 5: Matrices

Using row reduction to find the inverse – 3×3 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 8 & 2 & 2 \\ 4 & 8 & 10 \end{pmatrix}$$

Augmenting the matrix, we have:

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 7 & 1 & 0 & 0 \\ 8 & 2 & 2 & 0 & 1 & 0 \\ 4 & 8 & 10 & 0 & 0 & 1 \end{array}\right).$$

Chapter 5: Matrices

Using row reduction to find the inverse – 3×3 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 8 & 2 & 2 \\ 4 & 8 & 10 \end{pmatrix}$$

Again, we aim to have the left side become the identity matrix. So performing elementary row operations, we get:

$$A^{-1} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/9 & 1/9 & 1/18 \\ 0 & 1 & 0 & 2 & 1/2 & -3/2 \\ 0 & 0 & 1 & -14/9 & -4/9 & 23/18 \end{array}\right).$$

Chapter 5: Matrices

Using row reduction to find the inverse – 3×3 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 8 & 2 & 2 \\ 4 & 8 & 10 \end{pmatrix}$$

So the inverse of A is given by the matrix:

$$\begin{pmatrix} -1/9 & 1/9 & 1/18 \\ 2 & 1/2 & -3/2 \\ -14/9 & -4/9 & 23/18 \end{pmatrix}.$$

Chapter 5: Matrices

Using row reduction to find the inverse – 3×3 matrix

Find the inverse of A, where

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 8 & 2 & 2 \\ 4 & 8 & 10 \end{pmatrix}$$

So the inverse of A is given by the matrix:

$$\begin{pmatrix} -1/9 & 1/9 & 1/18 \\ 2 & 1/2 & -3/2 \\ -14/9 & -4/9 & 23/18 \end{pmatrix}.$$

ew!

Chapter 5: Matrices

Inverse of a matrix - example

MATH1131 2018 Semester 1 Q3iv

Given that the invertible $n \times n$ matrix A satisfies

$$A^2 = 2A + I,$$

express the inverse of A in terms of A and I.

Chapter 5: Matrices

Inverse of a matrix - example

MATH1131 2018 Semester 1 Q3iv

Given that the invertible $n \times n$ matrix A satisfies

$$A^2 = 2A + I,$$

express the inverse of A in terms of A and I.

Multiplying both sides by the inverse of A on the right side, we get:

$$\begin{aligned} A^2 A^{-1} &= (2A + I)A^{-1} \\ A &= 2AA^{-1} + A^{-1} \\ A - 2I &= A^{-1} \\ \therefore A^{-1} &= A - 2I. \end{aligned}$$

Chapter 5: Matrices

Inverse of a matrix - A final example

MATH1151 2016 Semester 1 Q3v

Prove that if an $n \times n$ matrix A is invertible and both A and A^{-1} have only integer entries then $\det(A) = \pm 1$.

Chapter 5: Matrices

Inverse of a matrix - A final example

MATH1151 2016 Semester 1 Q3v

Prove that if an $n \times n$ matrix A is invertible and both A and A^{-1} have only integer entries then $\det(A) = \pm 1$.

$$\begin{aligned} AA^{-1} = I &\implies \det\left(AA^{-1}\right) = 1 \\ &\implies \det(A)\det\left(A^{-1}\right) = 1 \\ &\implies \det(A) = \frac{1}{\det\left(A^{-1}\right)}. \end{aligned}$$

Now, since A and A^{-1} have only integer entries, then its determinants are integers. Since $\det(A)$ is an integer, then $\det\left(A^{-1}\right) = \pm 1$, which implies that $\det(A) = \pm 1$.

Good luck!!!

Best of luck with the exam! We hope that you found this seminar valuable!

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