UNSW Mathematics Society

Discrete Mathematics

Seminar II / II

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Seminar Overview

Part I: Proofs and logic

Proof vs intuition

Intuition:

1. A "heuristic".
2. Gives us a good idea whether a result is correct or not.
3. Prone to bias errors.
4. Improved overtime as we *prove* results.

Proofs:

1. Heavy rigour but provides a solid foundation for intuition.
2. Results proven can often be generalised.

Quantifiers

Universal and existential quantifier

	Symbol	English equivalent	LaTeX code
Universal	$\forall$	For all	`\forall`
Existential	$\exists$	There exists	`\exists`

Propositions, connectives and compound propositions

- A **proposition** is a statement that may or may not be true. In this topic, we will determine the absolute truth of this statement.
- A **connective** is a symbol that "connects" two propositions together.
  - $\wedge$ = and.
  - $\vee$ = or.
  - $\neg = \sim$ = not.
- **Compound propositions** consist of more than one proposition.

Implications and biconditionals (iff.)

- An **implication** means that one statement can be inferred indirectly from a previous statement. A statement q that can be inferred from a statement p is written as

$$p \implies q.$$

- If $p \implies q$ **AND** $q \implies p$, then we can say that

$$p \iff q$$

or p *if and only if* q.

Truth tables

A **truth table** is a table of "truth values" that allows us to determine the truth about a proposition.

p	q	$p \vee q$	$p \wedge q$	$p \implies q$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

Question: What happens if the statement is:

- **always** true?
- **always** false?
- **sometimes** true and **sometimes** false?

Truth tables – tautologies

If the statement is **always true**, we call that a *tautology*. To prove that a statement is a tautology, we show it through a **truth table**.

Example: (17S2, 3i)

Show, using truth tables, that $(p \to q) \vee (\sim p \to r)$ is a **tautology**.

p	q	r	$p \to q$	$\sim p \to r$	$(p \to q) \vee (\sim p \to r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	F	T

Truth tables – contradictions and contingencies

- If the statement is **always false**, we call it a *contradiction*. To prove that a statement is a contradiction, we show it through a *truth table*.
- If the statement depends on the propositions (sometimes true and sometimes false), we call it a *contingency*. To prove that a statement is a contingency, we show it through a *truth table*.

Truth tables – logical equivalence

If two statements yield in the **same** outputs for all propositions, we call them *logically equivalent*. We show this through a *truth table* or *standard logical equivalences*.

The **standard logical equivalences** work in pretty much the same way as set theory equivalences! A list of standard equivalences will be posted in a separate document after the stream!

Example: (17S2, 3ii)

Show, using standard logical equivalences, $(q \vee \sim r) \to p$ is logically equivalent to $(r \vee p) \wedge (q \to p)$.

Converse and contraposition

Converse of a statement – in logical statements

The **converse** of $p \Rightarrow q$ is $q \Rightarrow p$.

Example of a converse

The **converse** of *Since I own a cat, I own a pet* is *Since I own a pet, I own a cat*.

Contrapositive of a statement – in logical statements

The **contraposition** of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$.

Example of a contraposition

The **contraposition** of *Since I own a cat, I own a pet* is *Since I don't own a pet, I don't own a cat*.

Negating logical symbols

Symbol	Negation	Statement	Negation
$\wedge$	$\vee$	$p \wedge q$	$\neg p \vee \neg q$
$\vee$	$\wedge$	$p \vee q$	$\neg p \wedge \neg q$
$\neg$		$\neg p$	p

Negating quantifiers

Negating $\forall$ statements

The **negation** of $\forall$ is $\exists$.

Negating $\exists$ statements

The **negation** of $\exists$ is $\forall$.

Example

Negate $\forall x \in \mathbb{R},\ \exists y \in \mathbb{C}$.

1. The negation of $\forall x \in \mathbb{R}$ is $\exists x \in \mathbb{R}$.
2. The negation of $\exists y \in \mathbb{C}$ is $\forall y \in \mathbb{C}$.

Negating quantifiers

Example: Negation of quantifiers (2018S2, Q3iv)

Let $a_1, a_2, a_3, \dots$ be a sequence of real numbers. The definition of the *limit* of the sequence, $\lim_{n\to\infty} a_n = \ell$, is

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} : \forall n \geq N \quad |a_n - \ell| < \epsilon. \tag{*}$$

a) Write in symbolic form the negation of (*), and simplify your answer so that the negation symbol is not used.

$$\exists \epsilon > 0 \;\; \forall N \in \mathbb{N} : \exists n \geq N \quad |a_n - \ell| \geq \epsilon.$$

Negating statements

Negating an implication

Recall that an implication $p \Rightarrow q$ is logically equivalent to $\neg p \vee q$. So the **negation** of $p \Rightarrow q$ is

$$p \wedge \neg q.$$

Example: Negating an implication (2016S1, Q3iv)

A function f defined on the open interval $D = (a, b)$ is called **uniformly continuous on** D if and only if

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall x_0 \in D \ \ \forall x \in D :$$
$$|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon.$$

Write down the negation of this definition, simplified so that it does not contain the "not" symbol.

Rules of inference I

Modus arguments (rules of inference)

- **Modus ponens**
  - if $P \Rightarrow Q$ and P, then Q.
  - **Example**: If today is Tuesday, then I will go to work. Today is Tuesday. Therefore, I will go to work.
- **Modus tollens**
  - if $P \Rightarrow Q$ and $\neg Q$, then $\neg P$.
  - **Example**: If today is Tuesday, then I will go to work. I'm not going to work. Therefore, today is not Tuesday.

Rules of inference II

Syllogism arguments (rules of inference)

- **Hypothetical syllogism**
  - if $P \Rightarrow Q$ and $Q \Rightarrow R$, then $P \Rightarrow R$.
  - **Example**: If I do not wake up, then I will not go to work. If I do not go to work, I will not get paid. Therefore, if I do not wake up, I will not get paid.
- **Disjunctive syllogism**
  - if $P \vee Q$ and $\neg P$, then Q.
  - **Example**: I either study maths or computer science and I do not study maths. Therefore, I study computer science.

Rules of inference III

To prove that an argument is **valid**, we use a *truth table*.

- Reject any row that *denies* the hypotheses.
- Determine if the rest of the rows is true or not.

Example: (19T2, Q3ii)

Consider the following argument. "If I buy a new car then I will have to give up eating out and seeing movies. If I have to give up eating out then I won't give up seeing movies. Therefore, I won't buy a new car."
Show that the argument is **logically valid**.

Rules of inference III

Define $p =$ buy a new car, $q =$ give up eating and $r =$ give up seeing movies.
Construct the hypotheses and conclusion.

- **Hypotheses**:

$$p \implies q \wedge r.$$
$$q \implies \sim r.$$

- **Conclusion**:

$$\sim p.$$

Rules of inference III

Construct the truth table.

p	q	r	$p \to q \wedge r$	$q \to \sim r$	$\sim p$
T	T	T	T	F	$*$
T	T	F	F	$*$	$*$
T	F	T	F	$*$	$*$
T	F	F	F	$*$	$*$
F	T	T	T	F	$*$
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Proof writing

Tips for writing a good proof

1. State all of the assumptions you are making within the proof.
    - *Suppose that x is odd...*
    - *Suppose that y is prime...*
2. State all of the axioms and/or theorems that you will use throughout the proof.
    - *By De Moivre's theorem,* $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.
3. Write in **concise** and **full sentences**.
    - Avoid using logical statements inline with text: *Then it follows that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that ...*
    - Instead, consider writing in **prose**: *Then it follows that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that...*

Proof I: Direct proofs I

Process of proof

1. Begin with the hypothesis (if given).
2. Use some logical and deductive reasoning to reach the conclusion.

Example: Direct proofs

Prove that, if n is odd, then n^2 is also odd.

Proof II: Mathematical induction

Process of proof

1. Show that the statement holds for the **base case**.
2. (**Inductive hypothesis**) Assume that the statement holds for some integer $(n = k)$.
3. Show that the statement holds for the next case along $(n = k + 1)$.

Proof II: Mathematical induction

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Example: (16S2, Q3ii)

- Prove that $x^{n+1} - y^{n+1} = (x + y)(x^n - y^n) - xy(x^{n-1} - y^{n-1})$.

Proof II: Mathematical induction

Process of proof

1. Show that the statement holds for the **base case**.
2. (**Inductive hypothesis**) Assume that the statement holds for some integer $(n = k)$.
3. Show that the statement holds for the next case along $(n = k + 1)$.

Example: (16S2, Q3ii)

- Let $\alpha = 1 + \sqrt{5}$ and $\beta = 1 - \sqrt{5}$. Use mathematical induction to prove that

$$F_n = \frac{\alpha^n - \beta^n}{2^n \sqrt{5}}$$

is an integer for $n = 1, 2, 3, \ldots$.

Proof II: Mathematical induction

Process of proof

1. Show that the statement holds for the **base case**.
2. (**Inductive hypothesis**) Assume that the statement holds for some integer $(n = k)$.
3. Show that the statement holds for the next case along $(n = k + 1)$.

Example: (Tutorial Q53)

Prove that for all $n \in \mathbb{Z}^+$,

$$21 \mid 4^{n+1} + 5^{2n-1}.$$

Proof II: Mathematical induction

Process of proof

1. Show that the statement holds for the **base case**.
2. (**Inductive hypothesis**) Assume that the statement holds for some integer $(n = k)$.
3. Show that the statement holds for the next case along $(n = k + 1)$.

Example: (18S2, Q3iii)

Prove by mathematical induction that for all integers $n \geq 2$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

Proof III: Proof by contradiction

Process of proof

1. Assume that a statement is true.
2. Use mathematical deduction to arrive at a **contradiction**.
3. Conclude that the original statement must have been false.

Example: (17S2, Q3iii)

Prove that $\sqrt{13}$ is irrational.

Proof IV: Proof by contraposition

Process of proof

1. Rewrite the original statement into its **contrapositive**

$$p \implies q \iff \neg q \implies \neg p.$$

2. Prove the contraposition.
3. Since the original statement is equivalent to its contraposition, then the original statement must also be true.

Proof V: Proof by pigeonhole principle I

- (16S1) Suppose that 26 integers are chosen from the set $S = \{1, 2, \ldots, 50\}$. By writing these numbers as $2^k m$ with m odd, prove that one of the chosen numbers is a multiple of another of the chosen numbers.
- (17S1) Prove that given any 7 points on a circle of radius 1, there exist at least two that are less than 1 unit away from each other.
- (19T1) Let $b_1, b_2, \ldots, b_{14}$ be integers, with repetitions allowed. Define

$$S = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i, j \leq 14,\ i < j\}.$$

Prove that, for some $r \in \{0, 1, \ldots, 44\}$, there exist at least *three* pairs $(i, j) \in S$ such that

$$b_i + b_j \equiv r \pmod{45}.$$

Proof V: Proof by pigeonhole principle II

Process of proof

1. Reframe the question so you have *pigeons* and *pigeonholes*.
2. Use the pigeonhole principle to then finish the proof.

Proof VI: Mix of proofs

Often, you will need to use ideas taught in separate topics.

- (18S2) For all integers n, prove that 9 does not divide $n^2 - 3$.
- (16S1) If p and q are distinct primes, then $\sqrt{pq}$ is irrational.

Summary of proofs

- **Direct proofs**:
  - Key words: N/A
- **Mathematical induction**:
  - Key words: For all integers...
- **Proof by contradiction**:
  - Key words: Irrational, *does not*
- **Proof by contraposition**:
  - If the contraposition is easier to prove, use proof by contraposition.
- **Proof by pigeonhole principle**:
  - Key words: there exist *at least* X...

Part II: Enumeration and probability

Discrete Probability

Discrete Probability

- **Discrete probability** describes probabilities of outcomes where both the number of favourable outcomes and total outcomes are "countable".

$$Pr(\text{Event}) = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}$$

- We use various **counting** techniques to evaluate these probabilities.

Example: Discrete Probability

What is the probability of randomly picking a red marble out of a bag containing four red and six blue marbles?

Unconditional Counting I

Ordered Selections- No Repetition

- How many ways are there to choose r objects, in order, out of n "unique" objects, without replacement?
- Each of these selections is called a **permutation**. The total number of permutations is given by

$$P(n, r) = \frac{n!}{(n-r)!}$$

- Where r = n, the number of ways is given by the **factorial** of n:

$$n! = n \times (n-1) \times ... \times 2 \times 1$$

Unconditional Counting II

Unordered Selections- No Repetition

- How many ways are there to choose r objects, regardless of order, out of n "unique objects", without replacement?
- Each unordered selection is called a **combination**. The total number of combinations is given by

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Unconditional Counting III

Ordered Selections- Repetition Allowed

- How many ways are there to choose r objects, in order, out of$n$ "unique objects", with replacement?
- Since there are r independent spaces for n objects each, the number of choices is given by

$$n \times n \times ... \times n \text{ (r times)} = n^r$$

Unconditional Counting IV

Unordered Selections- Repetition Allowed

- How many ways are there to choose r objects, in any order, out of n "unique objects", with replacement?
- The total number of ways is given by

$$\binom{n+r-1}{n-1}$$

- Think of it as distributing r selections among n categories, using the "stars and bars" method.

Unconditional Counting V

Remember

- Check if there is replacement or not.
- Check if order is relevant.
- Check if the question asks for a probability.

A Useful Table

		Repetitions	
		allowed	not allowed
Order	relevant	n^r	$P(n, r)$
	irrelevant	$\binom{n+r-1}{n-1}$	$\binom{n}{r}$

Example I

Term 2 2019, Q2 (iii)

A poker hand consists of five cards dealt from a standard pack. A poker hand is called "four of a kind" if it contains four cards of the same value and one other card. You and one other player are each dealt a poker hand from the same pack. (So, ten cards are dealt altogether.)

a) What is the probability that your hand is "four of a kind"?

b) You pick up your hand and see that you have "four of a kind". What is the probability that the other player also has "four of a kind" ?

Remember

$$\text{Probability} = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}$$

Example I

a) **Solution:**

Total number of hands: $C(52,5)$

Number of ways to choose four cards of the same value:

$$C(13,1) = 13$$

Number of ways to get "four of a kind":

$$13 \times (52-4) = 13 \times 48$$

Probability:

$$\frac{13 \times 48}{C(52,5)} = \frac{1}{4165}$$

An alternative solution:

$$\frac{52 \times 12}{C(52,5)}$$

Example I

b **Solution:**
Total number of possible hands: $C(52-5,5) = C(47,5)$
Number of ways to choose four cards of the same value:

$$C(11,1) = 11$$

Number of ways to get "four of a kind":

$$11 \times (52-5-4) = 11 \times 43$$

Probability:

$$\frac{11 \times 43}{C(47,5)} = \frac{1}{3243}$$

Example II

Term 1 2019, Q2 (i)

The English alphabet has 26 letters, of which 5 are vowels and 21 are consonants. We will write all our words using upper case (capital) letters. Repetition of letters in words is allowed. Find the number of 12 letter words (strings) using the English alphabet:

a) with no further restrictions;
b) containing exactly 4 vowels and no repeated consonants;
c) containing the subword "WATER" at least once (e.g., "ZWATERZWATER" but not "ZWAZTERZZZZZ").

a) **Solution:**

$$26^{12}$$

Example II

b) **Solution:**

Choose the places for the vowels: $C(12, 4)$

Select the vowels (ordered, repetition allowed):

$$5^4$$

Select the consonants (ordered, no repetition):

$$P(21, 8)$$

Total words:

$$C(12, 4) \times 5^4 \times P(21, 8)$$

Multiplication/Addition Rule I

Multiplication Rule

- Two events are **independent** if the occurrence of one does not affect the probability of the other.

  A and B are independent $\iff Pr(\underline{\text{A and B}}) = Pr(A) \times Pr(B)$

- This is the **multiplication rule** and can only be applied to independent events.

Multiplication/Addition Rule II

Addition Rule

- Two items are **mutually exclusive** if they cannot both occur. i.e.

$$Pr(\text{A and B}) = 0$$

- The **addition rule** can always be applied where appropriate and is given by

$$Pr(\underline{\text{A or B}}) = Pr(A) + Pr(B) - Pr(\text{A and B})$$

More Complex Counting I

Grouped Objects

- When arranging objects, some objects may be required to stay together. To account for this, we count each of these groups as a single unit.
- Within each group, the objects may arranged in any order. We account for this by multiplying the total by the number of ways they can be rearranged.

Example III

Semester 2 2018, Q4 i)

How many different ways can 4 pairs of twins and 3 sets of triplets all from separate families be arranged in a line, if the siblings should stand together? (All 17 people can be distinguished from each other by their clothing.)

Solution:

Ways to arrange the line: 7!
Ways to arrange each of four pairs: 2!
Ways to arrange each of three sets of triplets: 3!
Multiplying them together gives a total of:

$$7! \times 2! \times 2! \times 2! \times 2! \times 3! \times 3! \times 3! = 7!(2!)^4(3!)^3$$

More Complex Counting II

Inclusion-Exclusion Principle

- The **inclusion-exclusion principle** revolves around the formula for counting elements in a union of sets.
- For a union of three sets, the total number of elements would be

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| =& |A_1| + |A_2| + |A_3| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ & + |A_1 \cap A_2 \cap A_3|\end{aligned}$$

Example II Part c

Term 1 2019, Q2 (i) c)

Find the number of 12 letter words (strings) using the English alphabet containing the subword "WATER" at least once (e.g.,"ZWATERZWATER" but not "ZWAZTERZZZZZ").

- c **Solution:**

  Containing at least 1 "WATER" (includes double counting)

  Choose the position of the "WATER":

  $$C(8,1) = 8$$

  Select the other letters (ordered, repetition allowed):

  $$26^7$$

  $$\text{Total ways} = 8 \times 26^7$$

Example II Part c

c) **Solution Continued:**
Containing 2 "WATER"s
Choose the position of the "WATER"s: $C(4,2) = 6$
Select the other letters:

$$26^2$$

$$\text{Total ways} = 6 \times 26^2$$

Number of words:

$$8 \times 26^7 - 6 \times 26^2$$

Example IV

Semester 2 2018, Q4 iii)

How many 13-card hands can be dealt from a standard deck of 52 cards such that

a) all cards are of the same colour?

b) there are exactly four cards in at least one suit?

a) **Solution:**

$$2 \times C(26, 13)$$

Alternatively,

$$\frac{52 \times C(25, 12)}{13}$$

Example IV

b) **Solution:**

Exactly four cards in at least one suit (includes double counting)

$$C(4,1) \times C(13,4) \times C(39,9)$$

Exactly four cards in at least two suits (includes double counting)

$$C(4,2) \times (C(13,4))^2 \times C(26,5)$$

Exactly four cards in three suits

$$C(4,3) \times (C(13,4))^3 \times C(13,1)$$

Number of ways:

$$\begin{aligned} &C(4,1) \times C(13,4) \times C(39,9) \\ &- C(4,2) \times (C(13,4))^2 \times C(26,5) \\ &+ C(4,3) \times (C(13,4))^3 \times C(13,1) \end{aligned}$$

More Complex Counting III

Repeated Objects

- The selection of objects to arrange may contain repetitions. Selecting any repeated item counts as the same selection, so the result is a lower number of arrangements.
- To eliminate the "double counting" of arranging repeated objects, divide the number of arrangements by the factorial of the number of each repetition. i.e.

$$\frac{n!}{n_A!n_B!n_C!\ldots}$$

- When selecting from a bag containing repetitions, the inclusion-exclusion principle will be required.

More Complex Counting IV

Stars and Bars

- How many ways are there to sort n objects into r baskets?
- We use n stars (*) to represent the n objects and r-1 bars (|) to represent dividers between baskets. e.g.

$$|***|**$$

- The number of ways to do this is given by:

$$C(n+r-1, r-1) = \binom{n+r-1}{r-1}$$

Example V

Term 2 2019, Q2 (iii)

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

where x_1, x_2, x_3, x_4, x_5 are to be non-negative integers.

a) How many solutions has this equation altogether?

b) How many solutions has this equation in which all x_1, x_2, x_3, x_4, x_5 are all congruent to 2 modulo 3?

c) How many solutions has this equation in which **none** of x_1, x_2, x_3, x_4, x_5 is congruent to 2 modulo 3?

Example V

a) **Solution:**
There are 100 units to be divided into 5 distinct groups $(x_1 \ldots)$. Thus the solution is

$$C(104, 4)$$

b) **Solution:**

$$x \equiv 2 \pmod 3 \iff x = 3n + 2 \text{ for some integer } n$$

Then we have,

$$3n_1 + 3n_2 + 3n_3 + 3n_4 + 3n_5 + 10 = 3 \times 30 + 10$$

$$n_1 + n_2 + n_3 + n_4 + n_5 = 30$$

The number of solutions is then:

$$C(34, 4)$$

Example V

c) **Solution:**

$$x_1 + x_2 + x_3 + x_4 + x_5 \equiv 100 \equiv 1 \pmod 3$$

Each term has two possibilities:

$$x_i \equiv 0 \pmod 3, \qquad x_i \equiv 1 \pmod 3$$

Case 1: (four terms $\equiv 0$)
Ways to choose the four terms: $C(5,4) = 5$

$$3n_1 + 3n_2 + 3n_3 + 3n_4 + 3n_5 + 1 = 3 \times 33 + 1$$

$$n_1 + n_2 + n_3 + n_4 + n_5 = 33$$

Number of ways: $5 \times C(37,4)$

Example V

c) **Solution Continued:**

Case 2: (one term $\equiv 0$)

Ways to choose one term: $C(5,1) = 5$

$$3n_1 + 3n_2 + 3n_3 + 3n_4 + 3n_5 + 4 = 3 \times 32 + 4$$

$$n_1 + n_2 + n_3 + n_4 + n_5 = 32$$

Number of ways: $5 \times C(36,4)$

Total

$$5 \times C(37,4) + 5 \times C(36,4)$$

More Complex Counting V

Binomials and Multinomials

- The coefficients of a **binomial** expansion under a non-negative integer power can be determined in the following way:

$$(x+y)^n = \sum_{r=0}^{n} C(n,r)x^r y^{n-r}$$

- A **multinomial** is similar to a binomial, except that it can have more than two terms inside the brackets.
- To determine the coefficient of a term in its expanded form, think of it as selecting from each of the n brackets.

$$(x+y+z)^n = (x+y+z) \times (x+y+z) \times \ldots \times (x+y+z)$$

$$\frac{n!}{n_x!n_y!n_z!} x^{n_x} y^{n_y} z^{n_z}$$

Example VI

Semester 2 2017, Q4 i)

Compute the coefficient of the monomial $x_1^3 x_2^2 x_3 x_4$ in the polynomial $(x_1 + x_2 + x_3 + x_4)^7$.

Solution:

$$\frac{7!}{3! \times 2! \times 1! \times 1!} = 420$$

More Complex Counting VI

Counting in a Circle

- Sometimes, a counting question would ask to arrange certain objects in a circle. To answer this type of question, divide the total number of arrangements by the number of "positions" in the circle.

More Complex Counting VII

Pigeonhole Principle

- The **pigeonhole principle** is a principle which uses counting techniques. The idea is that if n boxes contain more than n objects, then at least one box contains more than one object.
- Questions involving the use of the pigeonhole principle will often be in the form "what is the minimum number of people required such that there is at least r belonging to the same category?"
- The approach is to find the maximum number of people such that the condition is not met, and then add an extra person. If there are n categories, then the solution would be:

$$n \times (r - 1) + 1$$

Example VII

Semester 2 2016, Q4 v)

Suppose that 5 points are chosen in the plane. Each point has integer coordinates. Prove that the midpoint of the line segment joining at least two such points, also has integer coordinates.

$$(x_m, y_m) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

We know that $\underline{\text{odd} + \text{odd} = \text{even}}$ and $\underline{\text{even} + \text{even} = \text{even}}$.

Example VII

Solution

Each point has two coordinates (x, y) and each coordinate can either be even or odd. That gives four possible categories:

(even, even), (even, odd), (odd, even), (odd, odd)

With five such points, by the pigeonhole principle, there must be at least two points under the same category. From the information above, the midpoint between these two points will have integer coordinates.

Recurrence Relations I

Introduction

- A **recurrence relation** is a relation where a term in a sequence is defined based on previous terms. e.g.

$$F_n = F_{n-1} + F_{n-2}$$

- The relation also often comes with **initial conditions**.
- The goal is to be able to come up with an equation between each term and n.
- The methods to solving these involves a combination of guesswork and recognition.

Example VIII

Semester 2 2018, Q4 v)

A straight path of width 2 units is to be laid using the 1-unit by 2-unit paving slabs. [Such slabs can be laid side-by-side horizontally or vertically]

Let a_n be the number of ways to lay a path of width 2 units and length n units.

a) Find a_1, a_2 and a_3.

b) Obtain a recurrence relation for a_n. Explain your answer. (You do NOT need to solve this recurrence relation.)

Example VIII

a) **Solution:** Let | represent a vertical slab (1X2), **=** represent two horizontal slabs (2x2).

$$a_1 = 1$$

|

$$a_2 = 2$$

| | or **=**

$$a_3 = 3$$

| | | or |**=** or **=**|

Example VIII

b) **Solution:**
The last block is either a vertical slab (|) or a horizontal slab (=). Thus, the recurrence relation should be of degree 2.

There is one direct way to build a path of length n from a path of length n-2 (... =) and one way from a length of n-1 (... |). The recurrence relation is given by:

$$a_n = a_{n-1} + a_{n-2}$$

Recurrence Relations II

Solving Second Order Linear Recurrence Relations

- Start by rewriting all the terms containing a_n on one side and all other terms on the other:

$$a_n + pa_{n-1} + qa_{n-2} = f(n)$$

- Start by solving the **homogeneous case**. To do this, we "guess" that the solution will take the form $a_n = \lambda^n$
- Substituting this into the homogeneous equation gives:

$$\lambda^n + p\lambda^{n-1} + q\lambda^{n-2} = 0$$

- Factoring out λ^{n-2} (and the solution $\lambda = 0$) gives the **characteristic equation**:

$$\lambda^2 + p\lambda + q = 0$$

Recurrence Relations II

Solving Second Order Linear Recurrence Relations

- Where α and β are the solutions to the characteristic equation, the general solution to the recurrence relation will be:

$$a_n = A\alpha^n + B\beta^n \text{ (2 distinct, real roots)}$$

$$a_n = A\alpha^n + Bn\alpha^n \text{ (double root)}$$

$$a_n = \lambda^n(C\cos n\theta + D\sin n\theta) \text{ (2 non-real roots)}$$

- Substitute the **initial values** to find A and B if the problem is about a homogeneous recurrence.

Recurrence Relations III

Solving the Inhomogeneous Case

$$a_n + pa_{n-1} + qa_{n-2} = f(n)$$

- Once we have found the general solution to the homogeneous case, we then look for a **particular solution**. The solution to the inhomogeneous recurrence is then given by:

$$a_n = h_n + p_n$$

- The particular solution should have a similar form to the function on the right side of the equation.
- If the guess for the particular solution is part of the homogeneous solution, add an n in front.

Example IX

Semester 2 2018, Q4 iv)

a) Find the general solution of the recurrence

$$a_n + 4a_{n-1} - 12a_{n-2} = 0$$

subject to the conditions $a_0 = 3$ and $a_1 = 2$

b) Find the particular solution of the recurrence

$$a_n + 4a_{n-1} - 12a_{n-2} = 2^n$$

a) **Solution:**
Substitute $a_n = \lambda^n$

$$\lambda^n + 4\lambda^{n-1} - 12\lambda^{n-2} = 0$$

$$\lambda^2 + 4\lambda - 12 = 0$$

Example IX

a) **Solution Continued:**

$$a_n = A(-6)^n + B(2)^n$$

Substitute initial values:

$$A(-6)^0 + B(2)^0 = 3 \tag{1}$$

$$A(-6)^1 + B(2)^1 = 2 \tag{2}$$

$$A = \frac{1}{2} \, B = \frac{5}{2}$$

General (homogeneous) solution:

$$a_n = \frac{1}{2}(-6)^n + \frac{5}{2}(2)^n$$

Example IX

b **Solution:**

Substitute $a_n = Cn2^n$

$$Cn2^n + 4C(n-1)(2)^{n-1} - 12C(n-2)(2)^{n-2} = 2^n$$

$$(Cn + \frac{4C(n-1)}{2} - \frac{12C(n-2)}{2^2})2^n = 2^n$$

$$Cn + 2Cn - 2C - 3Cn + 6C = 1$$

$$C = \frac{1}{8}$$

Particular solution:

$$a_n = \frac{1}{8}n2^n$$