MATH3051: Applied Real and Functional Analysis

Lecture Notes by Gerald Huang

2023, Term 3

Contents

Chapter 1

Sets and Functions

1.1 Revision of Set Theory

1.2 Revision of Functions

1.3 Revision of Equivalence and Ordered Sets

1.4 Countability

Chapter 2

Metric Spaces

Our first object of study come from sets endowed with a notion of a *distance*. The ordinary distance function over the real numbers is the absolute difference; that is, over the real numbers, we can define the distance of x and y by the magnitude of its difference: $d(x, y) = |x - y|$. What we would like to do is to extract some of the useful properties from this distance function and generate a more general description of distance. What properties do we want in a distance function?

The first property that we would want is that the distance is non-negative; it doesn't *really* make sense to refer to negative distance. Therefore, one property that we would want is that, for any pair of points x, y in your set, $d(x, y) \geq 0$. What about the distance from x to itself? Define $d(x, x) = 0$ for each x.

The second property is that distances are symmetric; it takes the same distance if we started at x to y as it would if we started at y instead. Therefore, it would be nice if $d(x, y) = d(y, x)$. Finally, distances should satisfy the triangle inequality; that is, $d(x, y) \leq d(x, z) + d(z, y)$. In fact, this is enough to define a distance function, thereafter referred to as a *metric*.

Definition. Let X be a non-empty set. A *metric* on X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for any $x, y, z \in X$.

[1] **Positive definiteness**: $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$.

[2] **Symmetric**: $d(x, y) = d(y, x)$.

[3] **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric space*. These properties are clear for the metric $d(x, y) = |x - y|$ over the set $X = \mathbb{R}$. It is more interesting to explore more exotic examples of metric spaces.

Consider the set $C([a, b])$ of continuous functions on the closed interval $[a, b]$, and let $d_1(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$. It is not hard to see that $(C([a, b]), d_1)$ forms a metric space. The first two properties of the metric are obvious. It suffices to check the triangle inequality. We see that

$$\begin{aligned} d_1(f, g) &= \max_{a \leq t \leq b} |f(t) - g(t)| \\ &= \max_{a \leq t \leq b} |f(t) - h(t) + h(t) - g(t)| \\ &\leq \max_{a \leq t \leq b} (|f(t) - h(t)| + |h(t) - g(t)|) \\ &\leq \max_{a \leq t \leq b} |f(t) - h(t)| + \max_{a \leq t \leq b} |h(t) - g(t)| \\ &= d_1(f, h) + d_1(h, g), \end{aligned}$$

as required. Therefore, $(C([a, b]), d_1)$ forms a metric space.

2.1 Convergence in Metric Spaces

In previous calculus classes, we introduced the notion of *convergence* by looking at the limiting behaviour of a function. For example, $f(x) = 1/x$ converges to 0 as $x \to \infty$. More generally, the distance between $f(x)$ and 0 shrinks to zero as we take $x \to \infty$. This gives us the natural definition of *convergence* on a metric space.

Definition. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (X, d) is said to *converge* to $x \in X$ if for all $\epsilon > 0$, there exists $N > 0$ such that,

$$n > N \implies d(x_n, x) < \epsilon.$$

Therefore, a sequence $\{x_n\}$ may converge to x in a metric space (X, d_1) but not converge to x in another metric space (X, d_2), even when the underlying space is the same. As usual, there is a local notion of convergence – if $X = C([a, b])$, then $f_n(t) \to f$ as $n \to \infty$ for each $t \in [a, b]$. On the other hand, *uniform* convergence does not require us to fix each t; that is, for all $t \in [a, b]$, $f_n(t) \to f$.

2.1.1 Cauchy sequences

It is not hard to see that Cauchy sequences do not necessarily converge. Our goal is to find a requirement for convergence to be ensured in a Cauchy sequence. This will lead naturally to the concept of *completeness* which we explore further in Section 2.3.

Let $\{a_n\}$ be a Cauchy sequence in a metric space (X, d). It turns out that it is enough for a *subsequence* of $\{a_n\}$ to be convergent to ensure that $\{a_n\}$ also converges (not necessarily to a point in X).

Theorem. *Let $\{a_n\}$ be a Cauchy sequence in a metric space X. If $\{a_n\}$ has a convergent subsequence, then $\{a_n\}$ also converges to the same limit.*

Proof. Denote the subsequence by $\{a_{n_k}\}$ and suppose that $\{a_{n_k}\}$ converges to a. Our goal is to show that $\{a_n\}$ also converges to a. Since $\{a_n\}$ is Cauchy, then for each $\epsilon > 0$, there exists an N such that for all $n, m \geq N$, we have that

$$d(a_n, a_m) < \frac{\epsilon}{2}.$$

Since $\{a_{n_k}\}$ converges to a, then for each $\epsilon > 0$, there exists an N' such that for all $n_k \geq N'$, we have that

$$d(a_{n_k}, a) < \frac{\epsilon}{2}.$$

We now consider $d(a_n, a)$. Choose $M = \max\{N, N'\}$ so that both inequalities hold simultaneously. Then, for all $n \geq M$ and by the triangle inequality,

$$\begin{aligned} d(a_n, a) &\leq d(a_n, a_{n_k}) + d(a_{n_k}, a) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $\{a_n\} \to a$ which finishes the proof. □

In fact, the converse is also true and so, we have the following proposition.

Proposition. *A Cauchy sequence converges in a metric space if and only if it has a convergent subsequence.*

2.2 Continuity in Metric Spaces

2.3 Completeness

In the previous sections, we talked about sequences of points converging in a metric space. In particular, every convergence sequence is *Cauchy*. However, not all Cauchy sequences converge. The completeness property of metric spaces rectifies this.

Definition (Completeness)**.** A metric space X is *complete* if every Cauchy sequence converges to some element in X.

For example, the metric space $(\mathbb{Q}, |\cdot|)$ is *not* complete. Consider any irrational number x and consider the sequence given by the truncations of x. The sequence is Cauchy but does not converge to any element in $\mathbb{Q}$ because the sequence converges to x which is clearly not in $\mathbb{Q}$.

Theorem. *A closed subspace of a complete metric space is complete.*

Proof. Let X be a complete metric space, and let Y be a closed subspace of X. We now show that Y is complete. To do this, consider any Cauchy sequence $\{x_n\}$ in Y. We need to show that such a sequence converges to a point in Y. Clearly, $\{x_n\}$ is Cauchy in X. Since X is complete, $\{x_n\}$ converges to a point x in X. But Y is closed in X; therefore, Y must contain all of its limit points which implies that $x \in Y$ since $\{x_n\} \subseteq Y$. Therefore, $\{x_n\}$ converges to a point in Y which finishes the proof. □

2.3.1 Baire's Theorem

One reason why completeness is such a fundamental result is due to *Baire's theorem*. Loosely speaking, Baire's theorem says that if we begin with a complete metric space, then the intersection of every countable collection of dense open sets of X is dense in X. We can also think about this in the context of closed sets too. Given a collection of

2.4 Contraction Mappings

2.5 Applications

2.5.1 Existence and Uniqueness of solutions of ODEs

2.5.2 Picard-Lindelöf Theorem

2.5.3 Systems of Linear Equations

2.5.4 Fredholm Integral Equation

Chapter 3

Topological Spaces

In the previous chapter, we explored the concept of *metric spaces*. These were familiar objects to study but they aren't quite versatile. The metric object isn't quite general and so, some notions of convergence aren't well-defined on some set structures. When a set X does not have a metric equipped, we need to look for something stronger. It turns out that all we need is a notion of an *open set*.

In fact, metrics induce a very natural open set. Consider the metric defined by $d(x, y) = |x - y|$. One way to define an open set using d is to consider the collection of points y that is contained inside

3.1 Axioms of Separation

The axioms of separation impose further restrictions on a topological space (X, τ) to provide information about whether there are *enough* open sets to *separate* distinct points (subsets). The first natural restriction is the *existence* of an open set U that contains x but not y. Topological spaces that satisfy this property are often called T_0*-spaces* or *Kolmogorov spaces*. This is the weakest separation axiom but the simplest restriction to ensure that all points in a T_0-space are topologically distinguishable.

3.1.1 T_1-space

We can make this restriction stronger. Instead of requiring one open set that contains one point but not another, we might require a pair of open sets $U_x, V_y \in \tau$ such that $x \in U_x, y \notin U_x$ and $x \notin V_y, y \in V_y$ for all pairs of points $x, y \in X$. It is easy to see that all spaces that are T_1 must necessarily be T_0-spaces. These spaces are called *Fréchet spaces*.

3.1.2 T_2-space – Hausdorff property

We now come to an even stronger restriction. For T_1-spaces, the pair of open sets may overlap. The second axiom of separation requires the open sets to be disjoint; that is, for each pair of points $x, y \in X$, we have that

$$x \in U_x,\ y \in V_y,\ U_x \cap V_y = \emptyset.$$

These are called *Hausdorff spaces*.

3.2 Bases and Axioms of Countability

3.3 Convergence in Topological Spaces

3.4 Compactness

3.4.1 Compactness in Metric Spaces

3.5 Continuity in Topological Spaces

3.6 The Space $C(X)$

3.7 Connectedness

3.8 *The weak topology*

Chapter 4

Measure Theory and Lebesgue Integrals

Chapter 5

Normed Vector Spaces

Chapter 6

Differential Calculus in Normed Spaces